Chapter 11

Capital Asset Pricing Model (CAPM)

Road Map

Part A Introduction to finance.

Part B Valuation of assets, given discount rates.

Part C Determination of discount rates.

- Historic asset returns.
- Time value of money.
- Risk.
- Portfolio theory.
- Capital Asset Pricing Model (CAPM).
- Arbitrage Pricing Theory (APT).

Part D Introduction to corporate finance.

Main Issues

- Derivations of CAPM
- Implications of CAPM
- Empirical Evidence

Contents

1 Introduction

Portfolio theory analyzes investors' asset demand given asset returns.

This chapter studies how investors' asset demand determines the relation between assets' risk and return in a market equilibrium (when demand equals supply).

▶ A model to price risky assets.

Main question of this chapter

$$\mathrm{E}\left[\tilde{r}_i\right] = ?$$

Main points of portfolio theory

1. Diversify to eliminate non-systematic risk.
2. Hold only the risk-free asset and the tangent portfolio.
3. An asset's systematic risk is measured by contribution to the risk of the tangent portfolio – its beta β_{iT}.
4. An asset's risk premium is proportional to its systematic risk:

$$\bar{r}_i - r_F = \beta_{iT}\left(\bar{r}_T - r_F\right).$$

▶ Identifying the tangent portfolio gives a pricing model.

2 The Market Portfolio

Definition: The market portfolio is the portfolio of all risky assets traded in the market.

Definition: The market capitalization of an asset is its total market value.

Suppose there are a total of $i = 1, \ldots, n$ risky assets. Asset i's market capitalization is

$$\text{MCAP}_i = (\text{price per share})_i \times (\#\text{ of shares outstanding})_i.$$

The total market capitalization of all risky assets is

$$\text{MCAP}_M = \sum_{i=1}^{n} \text{MCAP}_i.$$

The market portfolio is the portfolio with weights in each risky asset i being

$$w_i = \frac{\text{MCAP}_i}{\sum_{j=1}^{n} \text{MCAP}_j} = \frac{\text{MCAP}_i}{\text{MCAP}_M}.$$

We denote the market portfolio by $\mathbf{w}_M$.

3 Derivation of CAPM

Assumptions for this chapter:

1. Investors agree on the distribution of asset returns.
2. Investors have the same fixed (static) investment horizon.
3. Investors hold efficient frontier portfolios.
4. There is a risk-free asset:
   - paying interest rate r_F
   - in zero net supply.
5. Demand of assets equals supply in equilibrium.

Implications:

1. Every investor puts their money into two pots:
   - the riskless asset
   - a *single* portfolio of risky assets – the tangent portfolio.
2. All investors hold the *risky* assets in same proportions
   - they hold the same risky portfolio, the tangent portfolio.
3. The tangent portfolio is the market portfolio.

3.1 A Numerical Illustration of CAPM

CAPM requires that in equilibrium total asset holdings of all investors must equal the total supply of assets.

We show this through the example below.

There are only three risky assets, A, B and C. Suppose that the tangent portfolio is

$$\mathbf{w}_T = (w_{\mathrm{A}}, w_{\mathrm{B}}, w_{\mathrm{C}}) = (0.25, 0.50, 0.25).$$

There are only three investors in the economy, 1, 2 and 3, with total wealth of 500, 1000, 1500 billion dollars, respectively. Their asset holdings (in billion dollars) are:

Investor	Riskless	A	B	C
1	100	100	200	100
2	200	200	400	200
3	-300	450	900	450
Total	0	750	1500	750

Claim:

> The market portfolio is the tangent portfolio:
>
> $$\mathbf{w}_M = \mathbf{w}_T.$$

In equilibrium, the total dollar holding of each asset must equal its market value:

$$\text{Market capitalization of A} = \$750 \text{ billion}$$

$$\text{Market capitalization of B} = \$1500 \text{ billion}$$

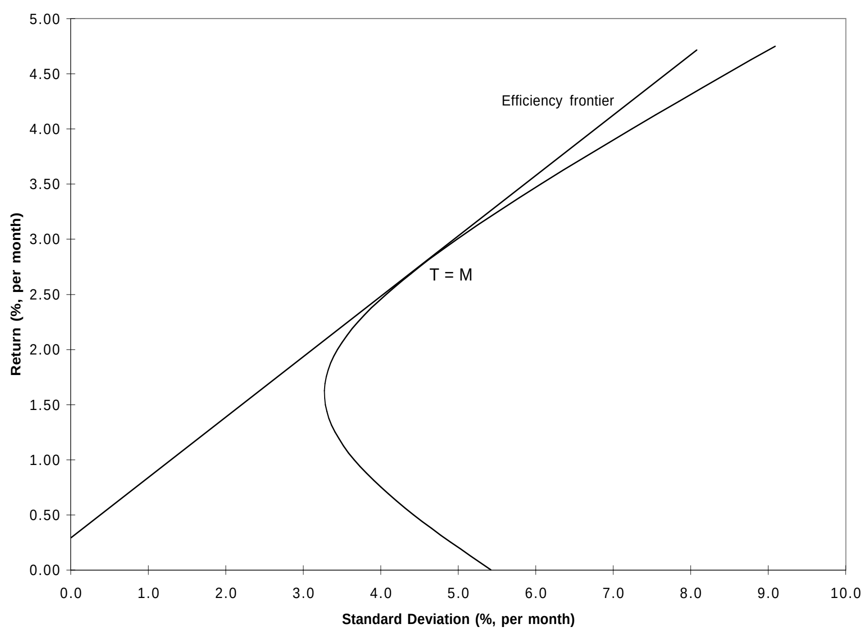
$$\text{Market capitalization of C} = \$750 \text{ billion.}$$

The total market capitalization is

$$750 + 1500 + 750 = \$3,000 \text{ billion.}$$

The market portfolio is the tangent portfolio:

$$\mathbf{w}_M = \left(\frac{750}{3000}, \frac{1500}{3000}, \frac{750}{3000}\right) = (0.25, 0.50, 0.25) = \mathbf{w}_T.$$



3.2 A Formal Derivation of CAPM

(1) There are $k = 1, 2, \ldots, K$ investors.

(2) Investor k has wealth W_k and invests in two funds:

- W_F^k in riskless asset
- $W^k - W_F^k$ in the tangent portfolio $\mathbf{w}_T$.

Market equilibrium — demand equals supply:

1. Money market equilibrium:

$$\sum_{k=1}^{K} W_F^k = 0 \quad \text{(risk-free asset is in zero net supply)}$$

2. Stock market equilibrium:

$$\sum_{k=1}^{K} \left(W^k - W_F^k\right) \mathbf{w}_T = \mathsf{MCAP}_M \mathbf{w}_M.$$

Since the net amount invested in the risk-free asset is zero, all wealth is invested in stocks:

$$\sum_{k=1}^{K} W^k = \mathsf{MCAP}_M.$$

Thus, the total wealth of investors equals the total value of stocks:

$$\mathbf{w}_T = \mathbf{w}_M.$$

The tangent portfolio is the market portfolio!

3.3 Implications of CAPM

1. The market portfolio is the tangent portfolio.

2. Combining the risk-free asset and the market portfolio gives the portfolio frontier.

3. The risk of an individual asset is characterized by its covariability with the market portfolio.

4. The part of the risk that is correlated with the market portfolio, the systematic risk, cannot be diversified away.
   - Bearing systematic risk needs to be rewarded.

5. The part of an asset's risk that is not correlated with the market portfolio, the non-systematic risk, can be diversified away by holding a frontier portfolio.
   - Bearing nonsystematic risk need not be rewarded.

6. For any asset i:

$$\boxed{\mathrm{E}[\tilde{r}_i] - r_F = \beta_{iM}\,(\mathrm{E}[\tilde{r}_M] - r_F) \qquad (11.1)}$$

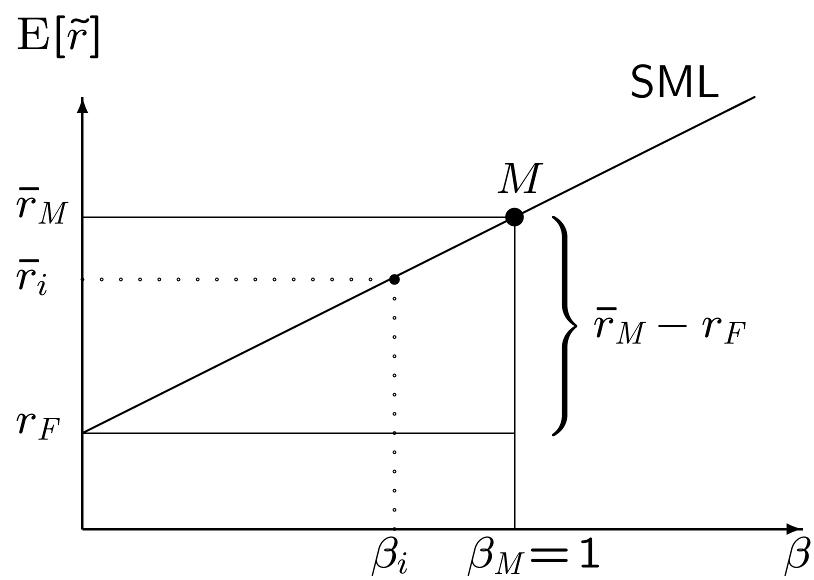
where $\beta_{iM} = \sigma_{iM}/\sigma^2_M$.

Given the premium of market portfolio, the riskless rate and assets' market betas, equation (11.1) determines the premium of all assets.

We thus have an asset pricing model — the CAPM.

The relation between an asset's risk premium and its market beta is called the "Security Market Line" (SML).

Security Market Line (SML)



Example. Suppose that CAPM holds. The expected market return is 14% and T-bill rate is 5%.

1. What should be the expected return on a stock with $\beta = 0$?

   Answer: Same as the risk-free rate, 5%. Note:

   - The stock may have significant uncertainty in its return.
   - This uncertainty is uncorrelated with the market return.

2. What should be the expected return on a stock with $\beta = 1$?

   Answer: The same as the market return, 14%.

3. What should be the expected return on a portfolio made up of 50% T-bills and 50% market portfolio?

   Answer: the expected return should be

   $\bar{r} = (0.5)(0.05) + (0.5)(0.14) = 9.5\%.$

4. What should be expected return on stock with $\beta = -0.6$?

   Answer: The expected return should be

   $\bar{r} = 0.05 + (-0.6)(0.14 - 0.05) = -0.4\%.$

   How can this be?

4 Understanding Risk in CAPM

In CAPM, we can decompose an asset's return into three pieces:

$$\tilde{r}_i - r_F = \alpha_i + \beta_{iM}(\tilde{r}_M - r_F) + \tilde{\varepsilon}_i$$

where

- $\mathrm{E}[\tilde{\varepsilon}_i] = 0$
- $\mathrm{Cov}[\tilde{r}_M, \tilde{\varepsilon}_i] = 0$.

Three characteristics of an asset:

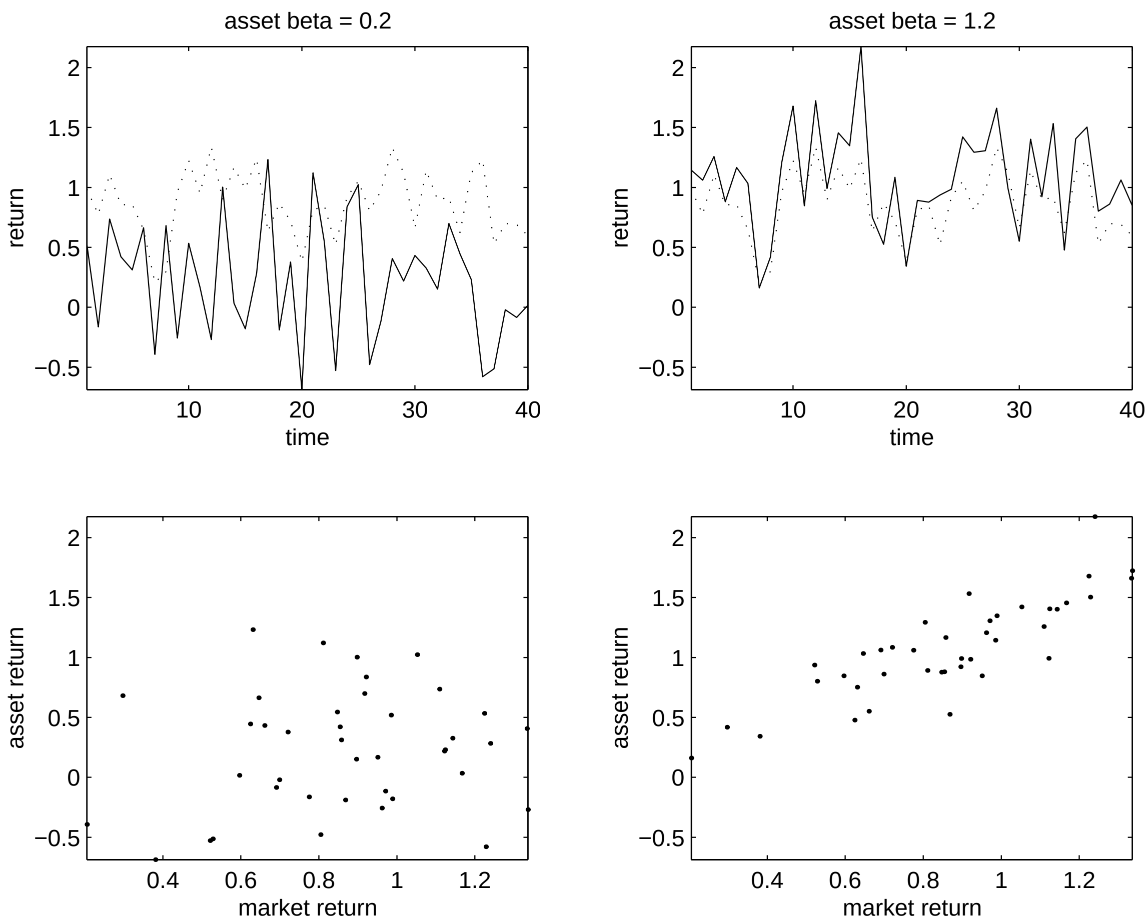
- Beta.
- Sigma = StD $(\tilde{\varepsilon}_i)$.
- Alpha.

Beta

$$\tilde{r}_i - r_F = \alpha_i + \boxed{\beta_{iM}} (\tilde{r}_M - r_F) + \tilde{\varepsilon}_i$$

- Beta measures an asset's systematic risk.
- Assets with higher betas are more sensitive to the market.

Two assets with same total volatility but different betas



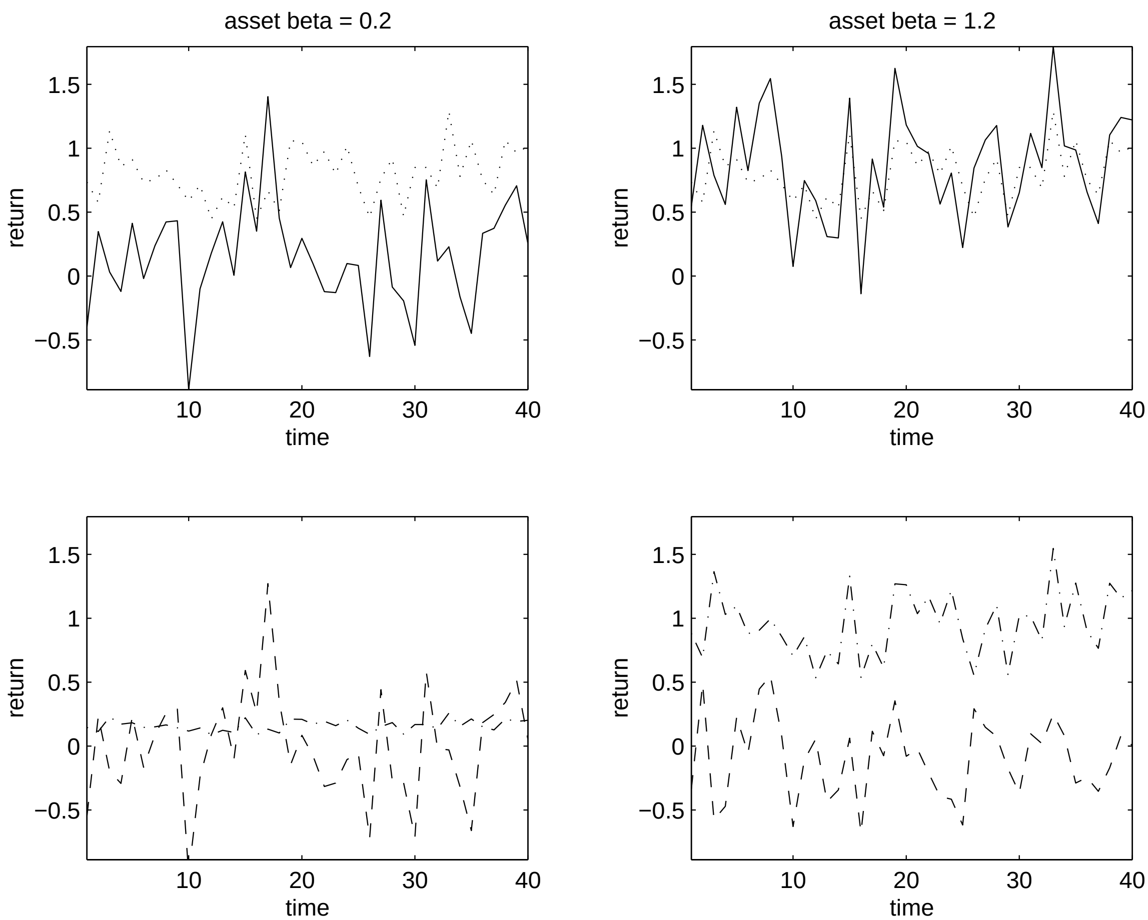
(Market premium = 8%, market volatility = 25%, asset volatility = 40%.)
Solid lines – asset rturns. Dotted lines – market returns.

Sigma

$$\tilde{r}_i - r_F = \alpha_i + \beta_{iM}(\tilde{r}_M - r_F) + \boxed{\tilde{\varepsilon}_i}$$

- An asset's sigma measures its non-systematic risk.
- Non-systematic risk is uncorrelated with systematic risk.

Two assets with same total volatility but different betas



(Market premium = 8%, market volatility = 25%, asset volatility = 40%.)
Solid lines – asset rturns. Dotted lines – market returns.
Dashdot lines – market component. Dashed lines – idiosyncratic component.

We can decompose return and risk as follows:

$$\tilde{r}_i - r_F = \overbrace{\beta_{iM}(\tilde{r}_M - r_F)}^{\substack{\text{systematic}\\ \text{component}}} + \overbrace{\tilde{\varepsilon}_i}^{\substack{\text{non-systematic}\\ \text{component}}} .$$

$$\overbrace{\mathsf{Var}[\tilde{r}_i]}^{\substack{\text{total}\\ \text{risk}}} = \overbrace{\beta_{iM}^2 \mathsf{Var}[\tilde{r}_M]}^{\substack{\text{systematic}\\ \text{risk}}} + \overbrace{\mathsf{Var}[\tilde{\varepsilon}_i]}^{\substack{\text{non-systematic}\\ \text{risk}}} .$$

Example. Systematic risk is only a part of return volatility. Consider an asset with

- annual volatility (σ) of 40%
- market beta of 1.2.

Suppose that the annual volatility of the market is 25%. What percentage of the total volatility of the asset is attributable to non-systematic risk?

$$\begin{aligned} (0.4)^2 &= (1.2)^2(0.25)^2 + \mathsf{Var}[\tilde{\varepsilon}]. \\ \mathsf{Var}[\tilde{\varepsilon}] &= 0.0700. \\ \sigma_{\varepsilon} &= 0.2645. \end{aligned}$$

$$\frac{\text{Non-systematic risk}}{\text{Total risk}} = \frac{0.07}{0.16} = 43.75\%.$$

Example. Two assets with the same *total* risk can have very different systematic risks.

Suppose that $\sigma_M = 20\%$.

Stock	Business	Market beta	Residual variance
1	Steel	1.5	0.10
2	Software	0.5	0.18

What is the total variance of each return?

$$\begin{aligned}\sigma_1^2 &= \beta_{1M}^2\sigma_M^2 + \sigma_{1\varepsilon}^2 \\ &= (1.5)^2(0.2)^2 + 0.10 \\ &= 0.19\end{aligned}$$

$$\begin{aligned}\sigma_2^2 &= \beta_{2M}^2\sigma_M^2 + \sigma_{2\varepsilon}^2 \\ &= (0.5)^2(0.2)^2 + 0.18 \\ &= 0.19.\end{aligned}$$

However

$$\begin{aligned}R_1^2 &= \frac{(1.5)^2(0.2)^2}{0.19} = 47\% \\ R_2^2 &= \frac{(0.5)^2(0.2)^2}{0.19} = 5\%.\end{aligned}$$

Alpha

$$\tilde{r}_i - r_F = \boxed{\alpha_i} + \beta_{iM}(\tilde{r}_M - r_F) + \tilde{\varepsilon}_i$$

- According to CAPM, α should be zero for all assets.
- α measures an asset's return in access of its risk-adjusted award according to CAPM.

What to do with an asset of positive α?

- Check estimation error.
- Past value of α may not predict its future value.
- Positive α may be compensating for other risks.
- ...

5 Applications of CAPM

Example. Required rates of return on IBM and Dell.

1. Use the value-weighted stock portfolio as a proxy for the market portfolio.

2. Regress historic returns of IBM and Dell on the returns on the value-weighted portfolio. Suppose the beta estimates are

   $$\beta_{\text{IBM,VW}} = 0.73 \quad \text{and} \quad \beta_{\text{Dell,VW}} = 1.63.$$

3. Use historic excess returns on the value weighted portfolio to estimated average market premium:

   $$\pi = \bar{r}_{\text{VW}} - r_F = 8.6\%.$$

4. Obtain the current riskless rate. Suppose it is

   $$r_F = 4\%.$$

5. Applying CAPM:

   $$\begin{aligned} \bar{r}_{\text{IBM}} &= r_F + \beta_{\text{IBM,VW}}(\bar{r}_{\text{VW}} - r_F) \\ &= 0.04 + (0.73)(0.086) = 0.1028. \end{aligned}$$

The expected rate of return on IBM (under CAPM) is 10.28%.

Similarly, the expected rate of return on Dell is 18.02%.

Reading the Beta Book

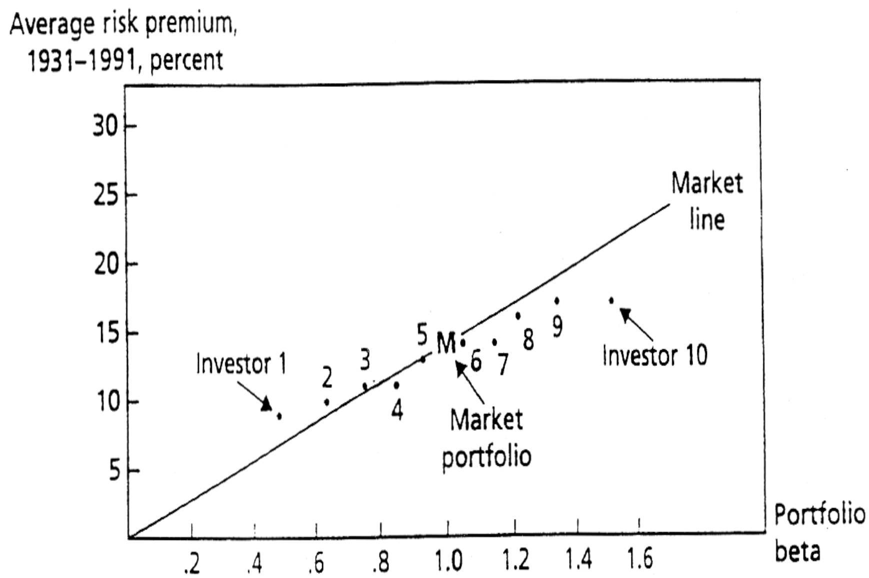
MLPF&S's Beta Book

Ticker Symbol		00/03 Close Price	Beta	Alpha	R-Sqr.	Resid Std Dev-N.	–Std. of Beta	Err.– of Alpha	Adj. Beta	Num. of Obs.
AALA	AmerAlia	2.250	-2.25	10.05	0.03	42.72	1.30	6.06	-1.15	60
AOL	America Online	67.438	2.40	4.12	0.25	17.17	0.52	2.44	1.93	60
GNLK	GeneLink	0.290	-7.74	16.15	0.44	37.30	2.10	9.92	-4.79	17
GM	General Mtrs	82.813	1.01	-0.57	0.27	6.98	0.21	0.99	1.01	60
TSN	Tyson Foods	11.125	1.13	-2.31	0.23	8.55	0.26	1.21	1.09	60

Note: (a) S&P 500 is used as a proxy for the market. (b) Betas are estimated with raw returns, not risk risk premiums. The alpha, according to CAPM, is $r_F(1-\beta)$. (c) Adjusted beta is obtained using other information.

6 Empirical Evaluation of CAPM

1. Long-run average returns are significantly related to beta:

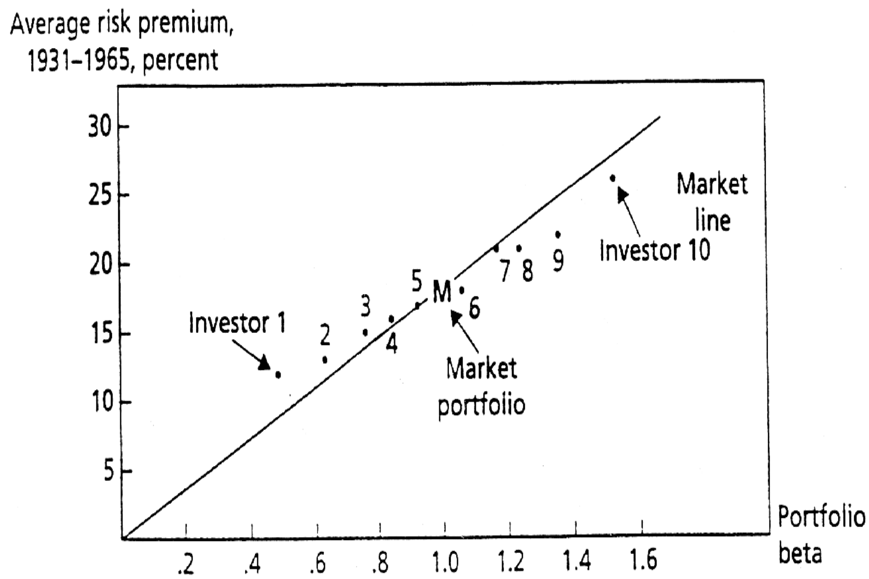


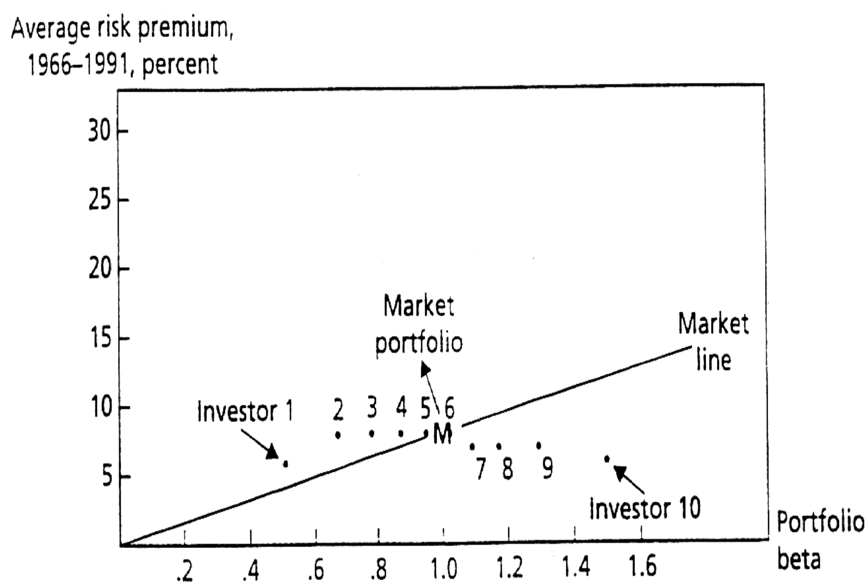
(Source: Fisher Black, "Beta and return.")

The dots show the actual average risk premiums from portfolios with different betas.

- high beta portfolios generated higher average returns
- high beta portfolios fall below SML
- low beta portfolios land above SML
- a line fitted to the 10 portfolios would be flatter than SML.

2. CAPM does not seem to work well over the last 30 years:

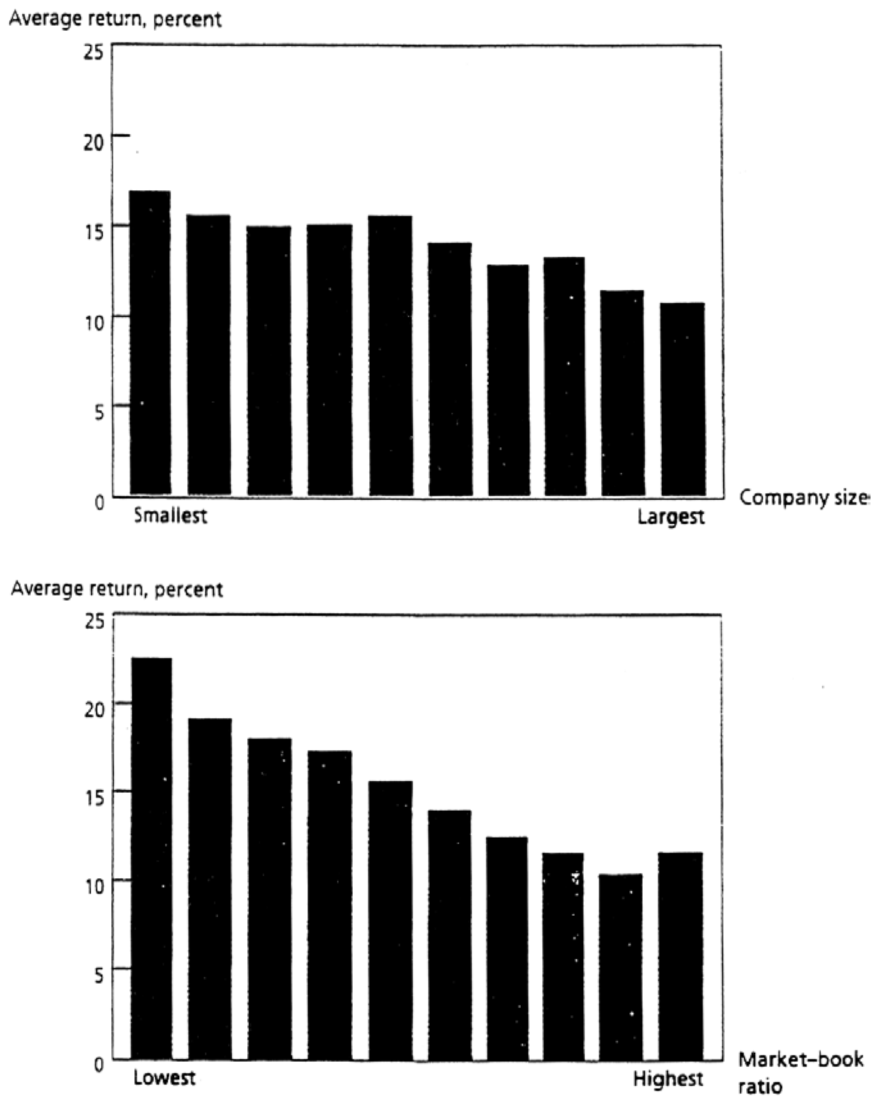




Source: Fischer Black, "Beta and return."

The dots show the actual average risk premiums from portfolios with different betas over different periods. The relation between beta and actual average return has been much weaker since the mid-1960s.

3. Factors other than beta seem important in pricing assets:



Source: G. Fama and K. French, "The Cross-Section of Expected Stock Returns".

Since mid-1960s:

- Small stocks have outperformed large stocks
- Stocks with low ratios of market-to-book value have outperformed stocks with high ratios.

7 Summary of CAPM

CAPM is *attractive*:

1. It is simple and sensible:
   - is built on modern portfolio theory
   - distinguishes systematic risk and non-systematic risk
   - provides a simple pricing model.
2. It is relatively easy to implement.

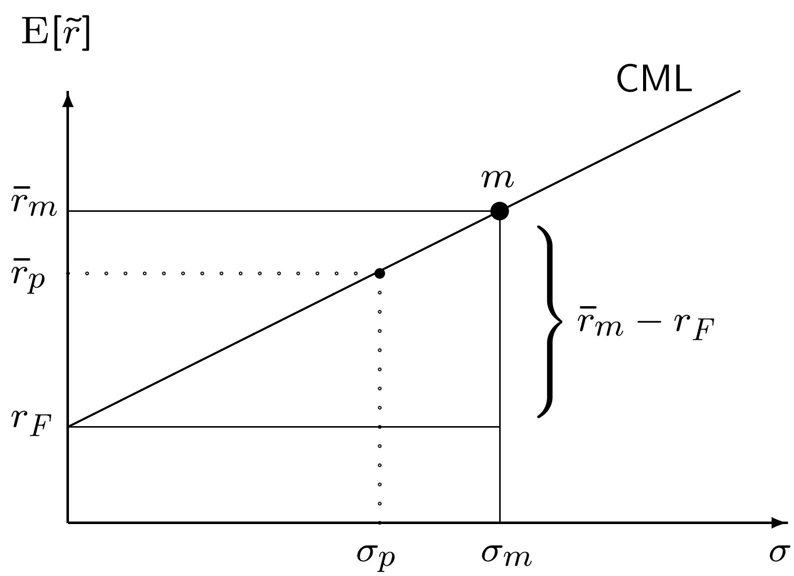
CAPM is *controversial*:

1. It is difficult to test:
   - difficult to identify the market portfolio
   - difficult to estimate returns and betas.
2. Empirical evidence is mixed.
3. Alternative pricing models might do better.
   - Multi-factor CAPM.
   - Consumption CAPM (C-CAPM).
   - APT.

8 Appendix A: Capital Market Line

In the presence of a risk-free asset, all efficient frontier portfolios lie on the "Capital Market Line" (CML):

Capital Market Line (CML)



- Investors hold only portfolios on CML
- The risk of an efficient frontier portfolio is its StD
- CML gives the trade-off between portfolio risk and return
- The price of one unit risk for an efficient portfolio is
  $$\frac{\mathrm{E}[\tilde{r}_m] - r_F}{\sigma_m}$$
  [the risk premium on market portfolio per unit of its StD].

9 Appendix B: Extensions of CAPM

9.1 Multifactor CAPM

In CAPM, investors care about returns on their investments over the next *short* horizon — they follow myopic investment strategies.

Myopic strategy is optimal if

1. Investors have only short horizons, or
2. Investors' asset demand does not change over time
   - future returns same as today:
     - the same r_F, $\bar{r}_m - r_F$, σ_m, β_{im} $(i = 1, \ldots, n)$
   - investors' risk preferences same as today.

Thus, even if investors actually invest for a long horizon, CAPM may work when future investment opportunities are the same as today.

In practice, however:

- Investors do invest over long horizons
- Investment opportunities do change over time.

Thus

- Investors may worry about unfavorable shifts in future investment opportunities.
- They may accept lower expected returns on assets that help to hedge against such shifts.

Example. Graduating MBAs saving for retirement may worry about a fall in future real interest rates.

- They might not be content with holding the market portfolio and the riskless asset.
- They might overweight assets that do well if real interest rates fall.
- They would be willing to accept low returns on these assets (relative to CAPM predictions).

How does this work?

1. In order to hedge the interest rate risk, construct a portfolio q that is uncorrelated with the market but highly correlated with changes in real interest rates.
2. The optimal portfolio now consists of three funds — "three-fund separation":
   - riskless asset
   - market portfolio
   - hedging portfolio q.
3. In equilibrium, an asset's premium is given by a multi-factor CAPM:
$$\bar{r}_i - r_F = \beta_{im}\left(\bar{r}_m - r_F\right) + \beta_{iq}\left(\bar{r}_q - r_F\right).$$

In general, there are two types of systematic risks:

1. Static (temporal) — Market risk
2. Dynamic (intertemporal) — Changes in investment opportunities.

Investors demand to be rewarded for bearing both types of risks.

Multi-factor CAPM tries to model the two types of systematic risk and how these risks are priced:

- Identify macroeconomic variables, the "factors", that might affect investment opportunities.

- Find portfolios of traded securities that are highly correlated with these factors.

- Hypothesize that the risk premium on an asset is linearly related to the risk premium on these portfolios:

$$\bar{r}_i - r_F = \alpha_i + \beta_{i1}\left(\bar{r}_{f1} - r_F\right) + \ldots + \beta_{iK}\left(\bar{r}_{fK} - r_F\right).$$

  where $\bar{r}_{fk}$ is the return on the portfolio that is correlated with only the k-th factor.

- The factor beta's can be estimated using regression analysis, as in the case of CAPM.

Advantages of multi-factor CAPM:

- Model captures different types of systematic risk

  (a) temporal

  (b) intertemporal.

- Model has the potential to fit data better.

Weaknesses of multi-factor CAPM:

- Model does may not identify the macroeconomic variables that constitute intertemporal risks.

- Model does may not specify the relative importance of these intertemporal risks.

9.2 Consumption CAPM (CCAPM)

To use the multifactor CAPM, we need to identify different sources of intertemporal risks in asset returns and specify their relative importance to investors. The theory itself gives little information on these factors and the data we have may not provide sufficient information either.

However, there is another way to characterize how investors perceive the risk in asset returns — a way that allows us to collapse many risk factors into one.

Consider a representative investor who decides between how much to invest in each asset:

She faces the following choices:

1. Consume \$1 today
   - achieve utility $u_0'(c_0)$, or
2. Invest the \$1 in asset i
   - receive $\$(1+\tilde{r}_i)$ tomorrow
   - consume the payoff, and achieve utility $u_1'(\tilde{c}_1)(1+\tilde{r}_i)$.

At optimum, she should be indifferent between choice 1 and 2.

- If 1 is better, sell asset i and consume today.
- If 2 is better, cut consumption today and invest in asset i.

Thus, at the optimum:

$$u_0'(c_0) = \mathrm{E}\left[u_1'(\tilde{c}_1)(1+\tilde{r}_i)\right].$$

In particular, for the risk-free asset:

$$u_0'(c_0) = \mathrm{E}\left[u_1'(\tilde{c}_1)(1+r_F)\right].$$

Take the difference between the two equations:

$$\mathrm{E}\left[u_1'(\tilde{c}_1)\,(\tilde{r}_i - r_F)\right] = 0.$$

Observation: For two random variables $\tilde{x}$ and $\tilde{y}$,

$$\mathrm{E}[\tilde{x}\tilde{y}] = \mathrm{Cov}[\tilde{x}, \tilde{y}] + \bar{x}\bar{y}.$$

Thus

$$\bar{r}_i - r_F = -\frac{1}{\mathrm{E}\left[u_1'(\tilde{c}_1)\right]}\mathrm{Cov}\left[u_1'(\tilde{c}_1), \tilde{r}_i\right].$$

An asset's risk premium is negatively related to the covariance between its return and an investor's future marginal utility.

Observation: The above must be true for any investor who is at optimum.

(a) For Cov $\left[u_1'(\tilde{c}_1), \tilde{r}_i\right] > 0$

- asset i pays when future marginal utility money is high
- its payoff is more valuable
- it commands negative premium.

(b) For Cov $\left[u_1'(\tilde{c}_1), \tilde{r}_i\right] < 0$

- asset i pays when future marginal utility of money is low
- its payoff is less valuable
- it commands positive premium.

Thus, an asset's risk is measured by the covariance between its return and investors' marginal utility.

More generally, we have

$$\mathrm{E_t}[\tilde{r}_{it+1}] - r_{Ft} = -\frac{1}{\mathrm{E_t}\left[u_1'(\tilde{c}_{t+1})\right]}\mathsf{Cov_t}\left[u_{t+1}'(\tilde{c}_{t+1}), \tilde{r}_{it+1}\right].$$

This is the Consumption CAPM.

Advantages of CCAPM:

- It gives the most general theory of risk.
- There is really only one risk — the risk in future consumption.

Weaknesses of CCAPM:

- Theory itself does not specify investors' marginal utilities.
- Implementation relies on consumption data, which is lacking.
- Theory relies on rationality of individual investors.

10 Homework

Readings:

- BKM Chapters 9.
- BM Chapters 8.2, 8.3.
- Readings package: "Beta and return" (F. Black).

Assignment:

- Problem Set 8.
- Project 2 on equity portfolio management.