GARP
DIGITAL LIBRARY

"The Capital Asset Pricing Model and Its Application to Performance Measurement"

Chapter 4 of

Portfolio Theory and Performance Analysis

By Noel Amenc and Veronique Le Sourd

4 The Capital Asset Pricing Model and its Application to Performance Measurement[1]

In Chapter 3 we described Markowitz's portfolio analysis model and presented the empirical market model. The latter was developed by Sharpe in order to simplify the calculations involved in the Markowitz model and thereby render it more operational. The next step in financial modelling was to study the influence of the behaviour of investors, taken as a whole, on asset prices. What resulted was a theory of asset valuation in an equilibrium situation, drawing together risk and return.

The model that was developed is called the Capital Asset Pricing Model (CAPM). Several authors have contributed to this model. Sharpe (1963, 1964) is considered to be the forerunner and received the Nobel Prize in 1990. Treynor (1961) independently developed a model that was quite similar to Sharpe's. Finally, Mossin (1966), Lintner (1965, 1969) and Black (1972) made contributions a few years later.

This model was the first to introduce the notion of risk into the valuation of assets. It evaluates the asset return in relation to the market return and the sensitivity of the security to the market. It is the source of the first risk-adjusted performance measures. Unlike the empirical market line model, the CAPM is based on a set of axioms and concepts that resulted from financial theory.

The first part of this chapter describes the successive stages that produced the model, together with the different versions of the model that were developed subsequently. The following sections discuss the use of the model in measuring portfolio performance.

4.1 THE CAPM

4.1.1 Context in which the model was developed

4.1.1.1 Investor behaviour when there is a risk-free asset

Markowitz studied the case of an investor who acted in isolation and only possessed risky assets. This investor constructs the risky assets' efficient frontier from forecasts on expected returns, variance and covariance, and then selects the optimal portfolio, which corresponds to his/her level of risk aversion on the frontier.

We always assume that the Markowitz assumptions are respected. Investors are therefore risk averse and seek to maximise the expected utility of their wealth at the end of the period. They choose their portfolios by considering the first two moments of the return distribution only, i.e. the expected return and the variance. They only consider one investment period and that period is the same for everyone.

[1] Numerous publications describe the CAPM and its application to performance measurement. Notable inclusions are Broquet and van den Berg (1992), Fabozzi (1995), Elton and Gruber (1995) and Farrell (1997).

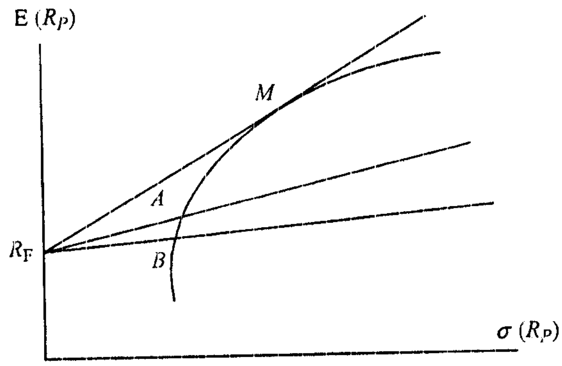

Figure 4.1 Construction of the efficient frontier in presence of a risk-free asset

Let us now consider a case where there is a risk-free asset. An asset is said to be risk-free when it allows a pre-determined level of income to be obtained with certainty. We shall write this asset's rate of return as R_F. Its risk is nil by definition. The investor can now spread his wealth between a portfolio of risky assets, from the efficient frontier, and this risk-free asset.

We take x to be the proportion of wealth invested in the risk-free asset. The remainder, or $(1 - x)$, is invested in the portfolio of risky assets, denoted as A. The expected return of the investor's portfolio P is obtained as a linear combination of the expected returns of its component parts, or

$$\mathrm{E}(R_P) = xR_F + (1 - x)\mathrm{E}(R_A)$$

and its risk is simply equal to

$$\sigma_P = (1 - x)\sigma_A$$

since the variance of the risk-free asset is nil and its covariance with the risky portfolio is also nil.

We can then eliminate x from the two equations and establish the following relationship:

$$\mathrm{E}(R_P) = R_F + \left(\frac{\mathrm{E}(R_A) - R_F}{\sigma_A}\right)\sigma_P \tag{4.1}$$

This is the equation of a straight line linking point R_F and point A. To be more explicit, let us see what happens graphically (see Figure 4.1). If we consider the representation of the Markowitz frontier on the plane $(\sigma_P, \mathrm{E}(R_P))$, the point corresponding to the risk-free asset is located on the y-axis. We can therefore trace straight lines from R_F that link up with the different points on the efficient frontier.[2] The equation of all these lines is equation (4.1). Among this set of lines there is one that dominates all the others and also dominates the frontier of risky assets at every point. This is the only line that forms a tangent with the efficient frontier. The point of tangent is denoted as M.

The $R_F M$ line represents all the linear combinations of the efficient portfolio of risky assets M with a risk-free investment. It characterises the efficient frontier in the case where one of

[2] We assume that the return R_F is lower than the return on the minimal variance portfolio (located at the summit of the hyperbola). Otherwise, the principle that a risky investment must procure higher revenue than a risk-free investment would not be respected.

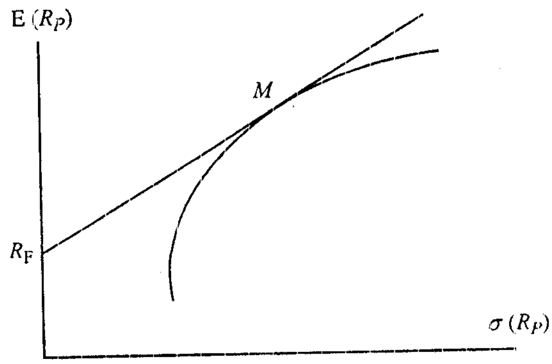

Figure 4.2 Efficient frontier in presence of a risk-free asset

the assets is risk-free. The introduction of a risk-free asset therefore simplifies the result, since the efficient frontier is now a straight line. In addition, the risk of the portfolios is reduced for a given return, since the straight line dominates the efficient frontier of risky assets at every point. Investors therefore benefit from having such an asset in their portfolio.

The choice of a particular portfolio on the line depends on the investor's level of risk aversion. The more risk averse the investor, the greater the proportion of the portfolio that he/she will invest in the risk-free asset. If the opposite is true, then the investor puts most of the portfolio into risky assets. Two cases are possible. (1) The investor has a limitless capacity to borrow, i.e. to invest negatively in the risk-free asset, in order to invest a sum that is greater than his wealth in risky assets. In this case, the efficient frontier is the line to the right of point M. (2) The borrowing is limited, in which case the efficient frontier is a straight line up to the point of tangency with the risky asset frontier and is then the curved portion of the risky asset frontier, since the segment of the line located above no longer corresponds to feasible portfolios (Figure 4.2).

The previous study assumed that the borrowing interest rate was equal to the lending interest rate. This assumes that the markets are frictionless, i.e. that the assets are infinitely divisible and that there are no taxes or transaction costs. This assumption will also be used in developing equilibrium theory.

It has therefore been established that when there is a risk-free asset, the investor's optimal portfolio P is always made up of portfolio M with x proportion of risky assets and proportion $(1 - x)$ of the risk-free asset. This shows that the investment decision can be divided into two parts: first, the choice of the optimal risky asset portfolio and secondly the choice of the split between the risk-free asset and the risky portfolio, depending on the desired level of risk. This result, which comes from Tobin (1958), is known as the two-fund separation theorem.

This theorem, and Black's theorem, which was mentioned in Chapter 3, have important consequences for fund management. Showing that all efficient portfolios can be written in the form of a combination of a limited number of portfolios or investment funds made up of available securities greatly simplifies the problem of portfolio selection. The problem of allocating the investor's wealth then comes down to the choice of a linear combination of mutual funds.

The position of the optimal risky asset portfolio M has been defined graphically. We now establish its composition by reasoning in terms of equilibrium.

4.1.1.2 Equilibrium theory

Up until now we have only considered the case of an isolated investor. By now assuming that all investors have the same expectations concerning assets, they all then have the same return, variance and covariance values and construct the same efficient frontier of risky assets. In the presence of a risk-free asset, the reasoning employed for one investor is applied to all investors. The latter therefore all choose to divide their investment between the risk-free asset and the same risky asset portfolio M.

Now, for the market to be at equilibrium, all the available assets must be held in portfolios. The risky asset portfolio M, in which all investors choose to have a share, must therefore contain all the assets traded on the market in proportion to their stock market capitalisation. This portfolio is therefore the market portfolio. This result comes from Fama (1970).

In the presence of a risky asset, the efficient frontier that is common to all investors is the straight line of the following equation:

$$\mathrm{E}(R_P) = R_\mathrm{F} + \left(\frac{\mathrm{E}(R_M) - R_\mathrm{F}}{\sigma_M}\right)\sigma_P$$

This line links the risk and return of efficient portfolios linearly. It is known as the capital market line.

These results, associated with the notion of equilibrium, will now allow us to establish a relationship for individual securities.

4.1.2 Presentation of the CAPM

We now come to the CAPM itself (cf. Briys and Viala, 1995, and Sharpe, 1964). This model will help us to define an appropriate measure of risk for individual assets, and also to evaluate their prices while taking the risk into account. This notion of the "price" of risk is one of the essential contributions of the model.

The development of the model required a certain number of assumptions. These involve the Markowitz model assumptions on the one hand and assumptions that are necessary for market equilibrium on the other. Some of these assumptions may seem unrealistic, but later versions of the model, which we shall present below, allowed them to be scaled down. All of the assumptions are included below.

4.1.2.1 CAPM assumptions[3]

The CAPM assumptions are sometimes described in detail in the literature, and sometimes not, depending on how the model is presented. Jensen (1972a) formulated the assumptions with precision. The main assumptions are as follows:

1. Investors are risk averse and seek to maximise the expected utility of their wealth at the end of the period.
2. When choosing their portfolios, investors only consider the first two moments of return distribution: the expected return and the variance.
3. Investors only consider one investment period and that period is the same for all investors.

[3] These assumptions are described well in Chapter 5 of Fabozzi (1995), in Cobbaut (1997), in Elton and Gruber (1995) and in Farrell (1997), who clearly distinguishes between the Markowitz assumptions and the additional assumptions.

4. Investors have a limitless capacity to borrow and lend at the risk-free rate.
5. Information is accessible cost-free and is available simultaneously to all investors. All investors therefore have the same forecast return, variance and covariance expectations for all assets.
6. Markets are perfect: there are no taxes and no transaction costs. All assets are traded and are infinitely divisible.

4.1.2.2 Demonstration of the CAPM

The demonstration chosen is the one given by Sharpe (1964). See also Poncet *et al.* (1996). It is the simplest and the most intuitive, since it is based on graphical considerations.[4]

We take the risk-free asset and the market portfolio. These two points define the capital market line. When the market is at equilibrium, the prices of assets adjust so that all assets will be held by investors: supply is then equal to demand. In theory, therefore, the market portfolio is made up of all traded assets, in proportion to their market capitalisation, even though in practice we use the return on a stock exchange index as an approximation of the market return.

We now take any risky asset i. Asset i is located below the market line, which represents all efficient portfolios.

We define a portfolio P with a proportion x invested in asset i and a proportion $(1-x)$ in the market portfolio. The expected return of portfolio P is given by

$$\mathrm{E}(R_P) = x\mathrm{E}(R_i) + (1-x)\mathrm{E}(R_M)$$

and its risk is given by

$$\sigma_P = \left[x^2\sigma_i^2 + (1-x)^2\sigma_M^2 + 2x(1-x)\sigma_{iM}\right]^{1/2}$$

where

σ_i^2 denotes the variance of the risky asset i;
σ_M^2 denotes the variance of the market portfolio; and
σ_{iM} denotes the covariance between asset i and the market portfolio.

By varying x, we construct the curve of all possible portfolios obtained by combining asset i and portfolio M. This curve goes through the two points i and M (see Figure 4.3).

The leading coefficient of the tangent to this curve at any point is given by

$$\frac{\partial \mathrm{E}(R_P)}{\partial \sigma_P} = \frac{\partial \mathrm{E}(R_P)/\partial x}{\partial \sigma_P/\partial x}$$

Now

$$\frac{\partial \mathrm{E}(R_P)}{\partial x} = \mathrm{E}(R_i) - \mathrm{E}(R_M)$$

and

$$\frac{\partial \sigma_P}{\partial x} = \frac{2x\sigma_i^2 - 2\sigma_M^2(1-x) + 2\sigma_{iM}(1-2x)}{2\sigma_P}$$

[4] The interested reader could refer to a more comprehensive demonstration in Chapter 9 of Briys and Viala (1995).

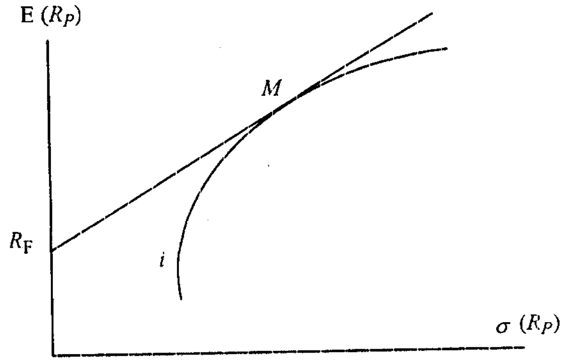

Figure 4.3 Curve of all portfolios made of the market portfolio and a risky asset

After simplifying, we obtain the following

$$\frac{\partial \mathrm{E}(R_P)}{\partial \sigma_P} = \frac{(\mathrm{E}(R_i) - \mathrm{E}(R_M))\sigma_P}{x\left(\sigma_i^2 + \sigma_M^2 - 2\sigma_{iM}\right) + \sigma_{iM} - \sigma_M^2}$$

The equilibrium market portfolio already contains asset i since it contains all assets. Portfolio P is therefore made up of an excess of asset i, in proportion x, compared with the market portfolio. Since this excess must be nil at equilibrium, point M is characterised by $x = 0$ and $\sigma_P = \sigma_M$.

When the market is at equilibrium, the slope of the tangent to the efficient frontier at point M is thus given by

$$\frac{\partial \mathrm{E}(R_P)}{\partial \sigma_P}(M) = \frac{(\mathrm{E}(R_i) - \mathrm{E}(R_M))\sigma_M}{\sigma_{iM} - \sigma_M^2}$$

Furthermore, the slope of the market line is given by

$$b = \frac{\mathrm{E}(R_M) - R_F}{\sigma_M}$$

where σ_M denotes the standard deviation of the market portfolio.

At point M the tangent to the curve must be equal to the slope of the market line. Hence, we deduce the following relationship:

$$\frac{(\mathrm{E}(R_i) - \mathrm{E}(R_M))\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\mathrm{E}(R_M) - R_F}{\sigma_M}$$

which can also be written as

$$\mathrm{E}(R_i) = R_F + \frac{(\mathrm{E}(R_M) - R_F)}{\sigma_M^2}\sigma_{iM}$$

The latter relationship characterises the CAPM. The line that is thereby defined is called the security market line. At equilibrium, all assets are located on this line.

This relationship means that at equilibrium the rate of return of every asset is equal to the rate of return of the risk-free asset plus a risk premium. The premium is equal to the price of the risk multiplied by the quantity of risk, using the CAPM terminology. The price of the risk is the difference between the expected rate of return for the market portfolio, and the return on the risk-free asset. The quantity of risk, which is called the beta, is defined by

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

Beta is therefore equal to the covariance between the return on asset i and the return on the market portfolio, divided by the variance of the market portfolio. The risk-free asset therefore has a beta of zero, and the market portfolio has a beta of one. The beta thus defined is the one that already appeared in Sharpe's empirical market model.

By using the beta expression, the CAPM relationship is then written as follows:

$$\mathrm{E}(R_i) = R_\mathrm{F} + \beta_i(\mathrm{E}(R_M) - R_\mathrm{F})$$

The CAPM has allowed us to establish that at equilibrium the returns on assets, less the risk-free rate, have a linear link to the return on the market portfolio, with the market portfolio being built according to Markowitz's principles.

This original version of the CAPM is based on assumptions that the financial markets do not completely respect. This first formula was followed by several other versions, which enabled the realities of the market to be taken into account to a greater degree. The different versions will be discussed in Section 4.1.3 below.

4.1.2.3 The contribution of the CAPM

The CAPM established a theory for valuing individual securities and contributed to a better understanding of market behaviour and how asset prices were fixed (cf. Chapter 3 of Farrell, 1997). The model highlighted the relationship between the risk and return of an asset and showed the importance of taking the risk into account. It allowed the correct measure of asset risk to be determined and provided an operational theory that allowed the return on an asset to be evaluated relative to the risk. The total risk of a security is broken down into two parts: the systematic risk, called the beta, which measures the variation of the asset in relation to market movements, and the unsystematic risk, which is unique for each asset. This breakdown could already be established with the help of the empirical market model, as we saw in Chapter 3. The unsystematic risk, which is also called the diversifiable risk, is not rewarded by the market. In fact, it can be eliminated by constructing diversified portfolios. The correct measure of risk for an individual asset is therefore the beta, and its reward is called the risk premium. The asset betas can be aggregated: the beta of a portfolio is obtained as a linear combination of the betas of the assets that make up the portfolio. According to the CAPM, the diversifiable risk component of each security is zero at equilibrium, while within the framework of the empirical market model only the average of the specific asset risks in the portfolio is nil.

The CAPM provides a reference for evaluating the relative attractiveness of securities by evaluating the price differentials compared with the equilibrium value. We should note that the individual assets are not on the efficient frontier, but they are all located on the same line at equilibrium. The CAPM theory also provided a context for developing manager performance evaluation, as we will show in Sections 4.2 and 4.3, by introducing the essential notion of risk-adjusted return.

By proposing an asset valuation model with the exclusive help of the market factor, Sharpe simplified the portfolio selection model considerably. He showed that optimal portfolios are obtained as a linear combination of the risk-free asset and the market portfolio, which, in practice, is approximated by a well-diversified portfolio. Equilibrium theory, which underlies the model, favoured the development of passive management and index funds, since it shows that the market portfolio is the optimal portfolio. The model also paved the way for the development of more elaborate models based on the use of several factors.

4.1.2.4 Market efficiency and market equilibrium

An equilibrium model can only exist in the context of market efficiency. Studying market efficiency enables the way in which prices of financial assets evolve towards their equilibrium value to be analysed. Let us first of all define market efficiency and its different forms.

The first definition of market efficiency was given by Fama (1970): markets are efficient if the prices of assets immediately reflect all available information. Jensen (1978) gave a more precise definition: in an efficient market, a forecast leads to zero profits, i.e. the expenses incurred in searching for information and putting the information to use offset the additional profit procured (cf. Hamon, 1997).

There are several degrees of market efficiency. Efficiency is said to be weak if the information only includes past prices; efficiency is semi-strong if the information also includes public information; efficiency is strong if all information, public and private, is included in the present prices of assets. Markets tend to respect the weak or semi-strong form of efficiency, but the CAPM's assumption of perfect markets refers in fact to the strong form.

The demonstration of the CAPM is based on the efficiency of the market portfolio at equilibrium. This efficiency is a consequence of the assumption that all investors make the same forecasts concerning the assets. They all construct the same efficient frontier of risky assets and choose to invest only in the efficient portfolios on this frontier. Since the market is the aggregation of the individual investors' portfolios, i.e. a set of efficient portfolios, the market portfolio is efficient.

In the absence of this assumption of homogeneous investor forecasts, we are no longer assured of the efficiency of the market portfolio, and consequently of the validity of the equilibrium model. The theory of market efficiency is therefore closely linked to that of the CAPM. It is not possible to test the validity of one without the other. This problem constitutes an important point in Roll's criticism of the model. We will come back to this in more detail at the end of the chapter.

The empirical tests of the CAPM involve verifying, from the empirical formulation of the market model, that the *ex-post* value of alpha is nil.

4.1.3 Modified versions of the CAPM[5]

Since the original assumptions of the CAPM are very restrictive, several authors have studied the consequences for the model of not respecting the assumptions. The studies address one assumption at a time. Chapter 14 of Elton and Gruber (1995) is very exhaustive on the subject. Among the versions of the model that were developed in this way, the most interesting from a practical application viewpoint are Black's zero-beta model and Brennan's model, which takes

[5] We can refer to Chapter 7 of Copeland and Weston (1988) for a detailed presentation with a demonstration of the Black and Merton models and to Chapter 8 for the Brennan model. Poncet *et al.* (1996) give a description without a demonstration of the Black and Merton models. Chapter 3 of Farrell (1997) presents the Black and Brennan models, and Chapter 14 of Elton and Gruber (1995) is devoted to the non-standard forms of the CAPM.

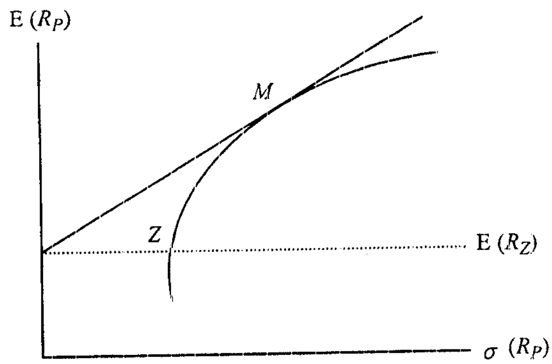

Figure 4.4 Minimum variance zero-beta portfolio (Z)

taxes into account. These two models will be shown in particular detail. This presentation is useful because it shows how the CAPM is adapted to the realities of the market. The different versions of the model are then applied in the area of portfolio performance measurement by allowing extensions to the Jensen measure. This will be developed in Section 4.2.

4.1.3.1 Black's zero-beta model[6]

Apart from the original version, this is the model that is most frequently used. This version was developed because two of the model's assumptions were called into question: (1) the existence of a risk-free asset, and therefore the possibility of borrowing or lending at that rate, and (2) the assumption of a single rate for borrowing and lending. Black (1972) showed that the CAPM theory was still valid without the existence of a risk-free asset, and developed a version of the model by replacing it with an asset or portfolio with a beta of zero. Instead of lending or borrowing at the risk-free rate, it is possible to take short positions on the risky assets.

The structure of the reasoning that enables this model to be produced is very close to that used to develop the basic model. We have the efficient frontier of risky assets, on which the market portfolio M is placed. We assume that we know how to determine the set of portfolios with zero beta, i.e. non-correlated with the market portfolio. These portfolios all have the same expected return $E(R_Z)$, since they all have the same systematic risk, namely a beta equal to zero. Among all these portfolios, only one is located on the efficient frontier: this is the portfolio with the minimum risk (see Figure 4.4).

We therefore have two portfolios on the efficient frontier: the market portfolio and the zero-beta portfolio, denoted by Z, with minimum variance. The complete efficient frontier can be obtained by combining these two portfolios. We invest x in portfolio Z and $(1-x)$ in portfolio M. The expected return of this portfolio is written as follows:

$$E(R_P) = xE(R_Z) + (1-x)E(R_M)$$

[6] For this model, we can also refer to Chapter 6 of Fabozzi (1995).

and its risk is

$$\sigma(R_P) = \left(x^2\sigma_Z^2 + (1-x)^2\sigma_M^2\right)^{1/2}$$

since the correlation between the market portfolio and portfolio Z is nil.

We then look for the slope of the tangent to point M that intersects the y-axis at point $E(R_Z)$. This slope is given by

$$\frac{\partial E(R_P)}{\partial\sigma(R_P)} = \frac{\partial E(R_P)/\partial x}{\partial\sigma(R_P)/\partial x}$$

We therefore calculate the partial derivatives of the expected return and risk of the portfolio, or:

$$\frac{\partial E(R_P)}{\partial x} = E(R_Z) - E(R_M)$$

and:

$$\frac{\partial\sigma(R_P)}{\partial x} = \frac{2x\sigma_Z^2 - 2\sigma_M^2(1-x)}{2\sigma(R_P)}$$

At point M, $x = 0$ and $\sigma(R_P) = \sigma_M$ so

$$\frac{\partial E(R_P)}{\partial\sigma(R_P)} = \frac{E(R_M) - E(R_Z)}{\sigma_M}$$

Furthermore, this line intersects the y-axis at the point $E(R_Z)$. Its equation is therefore finally written as follows:

$$E(R_P) = E(R_Z) + \left(\frac{E(R_M) - E(R_Z)}{\sigma_M}\right)\sigma(R_P)$$

The equation that is thereby established is identical in form to that of the capital market line of the basic model. The return on the risk-free asset is simply replaced by that of the zero-beta portfolio.

It is now possible to show that the return on any risky asset can be written using the return on the zero-beta portfolio and the return on the market portfolio. To do so, we proceed in the same way as when establishing the CAPM in the presence of a risk-free asset.

We consider the curve representing all the portfolios made up of a risky asset and the market portfolio. The slope of the tangent to this curve at point M is given by

$$\frac{(E(R_i) - E(R_M))\sigma_M}{\sigma_{iM} - \sigma_M^2}$$

This slope must be equal to the slope of our new market line, or

$$\frac{E(R_M) - E(R_Z)}{\sigma_M}$$

Hence

$$\frac{(E(R_i) - E(R_M))\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{E(R_M) - E(R_Z)}{\sigma_M}$$

which finally gives the following:

$$E(R_i) = E(R_Z) + \frac{\sigma_{iM}}{\sigma_M^2}(E(R_M) - E(R_Z))$$

or

$$E(R_i) = E(R_Z) + \beta_i(E(R_M) - E(R_Z))$$

since

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

The formula established is similar to that of the original CAPM, except that the return on the risk-free asset is replaced with the return on the zero-beta portfolio. The form of the CAPM is therefore conserved in the absence of a risk-free asset. This model is called the two-factor model.

Let us now return to the construction of zero-beta portfolios in more detail. A zero-beta portfolio is a portfolio with variations that are totally independent of market variations. We observe that most risky assets are positively correlated with each other. The best way to obtain a zero-beta portfolio is therefore to associate long and short positions on the assets, i.e. to carry out short selling on assets. The construction of zero-beta portfolios is not therefore possible unless short selling is authorised without any restrictions.

The CAPM cannot therefore be established without one or other of the following assumptions: the existence of a risk-free asset, which we can sell short without any limitations; or the absence of constraints on short selling of risky assets. It should be noted that generally there are restrictions on short selling. As a result, although this version of the model widens the framework for using the CAPM, it does not provide a solution in every case.

4.1.3.2 Model taking taxes into account: Brennan version

The basic CAPM model assumes that there are no taxes. The investor is therefore indifferent to receiving income as a dividend or a capital gain and all investors hold the same portfolio of risky assets. However, taxation of dividends and capital gains is generally different, and this is liable to influence the composition of the investors' portfolio of risky assets. Taking these taxes into account can therefore modify the equilibrium prices of the assets.

As a response to this problem, Brennan (1970) developed a version of the CAPM that allows the impact of taxes on the model to be taken into account. His model is formulated as follows:

$$E(R_i) = R_F + \beta_i(E(R_M) - R_F - T(D_M - R_F)) + T(D_i - R_F)$$

where

$$T = \frac{T_d - T_g}{1 - T_g}$$

and where

T_d denotes the average taxation rate for dividends;
T_g denotes the average taxation rate for capital gains;
D_M denotes the dividend yield of the market portfolio; and
D_i denotes the dividend yield of asset i.

By presenting this formula slightly differently, we have.

$$E(R_i) - R_F - T(D_i - R_F) = \beta_i(E(R_M) - R_F - T(D_M - R_F))$$

and we come back to a structure that is very similar to the basic CAPM. The returns on the asset and on the market are respectively decreased (or increased if T is negative) by a term proportional to the dividend yield and the taxes. When the tax rate on the dividends is equal to the tax rate on the capital gains, or $T = 0$, we do come back to the original model.

Investors can, for example, seek to avoid stocks that pay out large dividends. Such a strategy enables the return on the portfolio to be increased after deducting taxes, but, by distancing it from the market portfolio, it reintroduces a residual risk component.

4.1.3.3 Merton's continuous time version

Merton (1973) developed a continuous time version of the CAPM. His model is called the Intertemporal Capital Asset Pricing Model (ICAPM). In this model it is assumed that a state variable, for example the risk-free interest rate, evolves randomly over time. In this case, Merton shows that investors hold portfolios that result from three funds: the risk-free asset, the market portfolio and a third portfolio, chosen in such a way that its return is perfectly negatively correlated with the return on the risk-free asset. The two-fund separation model is replaced with a three-fund separation model. This third fund allows hedging against the risk of an unanticipated change in the future value of the risk-free rate (see also Elton and Gruber, 1995, and Copeland and Weston, 1998).

The expected return of an asset i at equilibrium is then written:

$$E(R_i) = R_F + \lambda_{i1}(E(R_M) - R_F) - \lambda_{i2}(E(R_{NF}) - R_F)$$

where

$$\lambda_{1i} = \frac{\beta_{iM} - \beta_{i,NF}\beta_{NF,M}}{1 - \rho^2_{NF,M}} \quad \text{and} \quad \lambda_{2i} = \frac{\beta_{i,NF} - \beta_{iM}\beta_{NF,M}}{1 - \rho^2_{NF,M}}$$

and where $\beta_{x,y}$ and $\rho_{NF,M}$ are defined as follows:

$$\beta_{x,y} = \frac{\sigma_{xy}}{\sigma^2_y} \quad \text{and} \quad \rho_{NF,M} = \frac{\sigma_{NF,M}}{\sigma_{NF}\sigma_M}$$

$E(R_{NF})$ denotes the expected rate of return of a portfolio that has perfect negative correlation with the risk-free asset R_F. All the rates of return are used in this model are continuous rates.

If the risk-free rate is not stochastic, or if it is not correlated with the market risk, then the third fund disappears, $\beta_{i,NF} = \beta_{NF,M} = 0$. We then come back to the standard formulation of the CAPM, except that the rates of return are instantaneous and the distribution of returns is lognormal instead of being normal.

The Merton model is a multi-period version of the CAPM. Assuming that the risk-free rate is stochastic leads to establishing a multi-factor (or multi-beta) version of the CAPM. Such a model can then be generalised to take other sources of extra-market risk into account, with the principle still being to make up a hedging portfolio for each source of risk and to determine the sensitivity of the assets to these portfolios. Nevertheless, this general theoretical approach does not specify the nature of the risk factors, or how to construct the portfolios to hedge the risks.

4.1.3.4 Model taking inflation into account

This model is a simple example of a generalisation of the CAPM for several factors. Here we assume that inflation is uncertain, which constitutes an additional risk factor on top of the basic model's market risk factor. The expected return at equilibrium of an asset i is now written as follows:

$$\mathrm{E}(R_i) - R_\mathrm{F} = \beta_{iM}(\mathrm{E}(R_M) - R_\mathrm{F}) + \beta_{iI}(\mathrm{E}(R_\mathrm{I}) - R_\mathrm{F})$$

where $\beta_{i\mathrm{I}}$ denotes the sensitivity of security i to the portfolio of securities held to hedge the inflation risk and $(\mathrm{E}(R_\mathrm{I}) - R_\mathrm{F})$ is the price of the inflation risk.

4.1.3.5 Model based on consumption: Breeden's (1979) CCAPM model

Here again we are dealing with a multi-period model, but one that is removed from the basic model since the returns on assets are no longer explained through the market return, but with the help of the consumption growth rate.

For each period t the return on asset i is written as follows:

$$R_{it} = \alpha_i + \beta_i C_t + e_{it}$$

where C_t denotes the consumption growth rate.

We also assume that the following conditions are respected:

$$\mathrm{E}(e_{it}) = 0$$
$$\mathrm{E}(e_{it}, C_t) = 0$$
$$\beta_i = \frac{\mathrm{cov}(R_{it}, C_t)}{\mathrm{var}(C_t)}$$

We can then establish the following equilibrium condition:

$$\mathrm{E}(R_i) = \mathrm{E}(R_Z) + \beta_i \gamma_1$$

where

γ_1 denotes the market remuneration for the consumption risk, with this risk being measured by the beta; and $\mathrm{E}(R_Z)$denotes the expected return on a zero-beta portfolio.

4.1.4 Conclusion

The presentation of some modified forms of the CAPM has allowed us to observe that the general structure of the basic model was quite well respected. We should however stress that these models were established by only modifying one assumption at a time. The advantage of these models is to be able to suggest improvements to the performance measurement indicators of the portfolios, while conserving a simple formula. We shall see the relevant applications in Section 4.2. The multi-factor forms of the model are already quite similar to the APT type models, to which Chapter 6 will be devoted.

4.2 APPLYING THE CAPM TO PERFORMANCE MEASUREMENT: SINGLE-INDEX PERFORMANCE MEASUREMENT INDICATORS[7]

When we presented the methods for calculating the return on a portfolio or investment fund in Chapter 2, we noted that the return value on its own was not a sufficient criterion for appreciating the performance and that it was necessary to associate a measure of the risk taken. Risk is an essential part of the investment. It can differ considerably from one portfolio to another. In addition, it is liable to evolve over time. Modern portfolio theory and the CAPM have established the link that exists between the risk and return of an investment quantitatively. More specifically, these theories highlighted the notion of rewarding risk. Therefore, we now possess the elements necessary for calculating indicators while taking both risk and return into account.

The first indicators developed came from portfolio theory and the CAPM. They are therefore more specifically related to equity portfolios. They enable a risk-adjusted performance value to be calculated. It is thus possible to compare the performance of funds with different levels of risk, while the return alone only enabled comparisons between funds with the same level of risk.

This section describes the different indicators and specifies, for each, their area of use. It again involves elementary measures because the risk is considered globally. We will see later on that the risk can be broken down into several areas, enabling a more thorough analysis.

4.2.1 The Treynor measure

The Treynor (1965) ratio is defined by

$$T_P = \frac{\mathrm{E}(R_P) - R_\mathrm{F}}{\beta_P}$$

where

$\mathrm{E}(R_P)$ denotes the expected return of the portfolio;
R_F denotes the return on the risk-free asset; and
β_P denotes the beta of the portfolio.

This indicator measures the relationship between the return on the portfolio, above the risk-free rate, and its systematic risk. This ratio is drawn directly from the CAPM. By rearranging the terms, the CAPM relationship for a portfolio is written as follows:

$$\frac{\mathrm{E}(R_P) - R_\mathrm{F}}{\beta_P} = \mathrm{E}(R_M) - R_\mathrm{F}$$

The term on the left is the Treynor ratio for the portfolio, and the term on the right can be seen as the Treynor ratio for the market portfolio, since the beta of the market portfolio is 1 by definition. Comparing the Treynor ratio for the portfolio with the Treynor ratio for the market portfolio enables us to check whether the portfolio risk is sufficiently rewarded.

The Treynor ratio is particularly appropriate for appreciating the performance of a well-diversified portfolio, since it only takes the systematic risk of the portfolio into account, i.e.

[7] On this subject, the interested reader could consult Broquet and van den Berg (1992), Elton and Gruber (1995), Fabozzi (1995), Grandin (1998), Jacquillat and Solnik (1997), and Gallais-Hamonno and Grandin (1999).

the share of the risk that is not eliminated by diversification. It is also for that reason that the Treynor ratio is the most appropriate indicator for evaluating the performance of a portfolio that only constitutes a part of the investor's assets. Since the investor has diversified his investments, the systematic risk of his portfolio is all that matters.

Calculating this indicator requires a reference index to be chosen to estimate the beta of the portfolio. The results can then depend heavily on that choice, a fact that has been criticised by Roll. We shall return to this point at the end of the chapter.

4.2.2 The Sharpe measure

Sharpe (1966) defined this ratio as the reward-to-variability ratio, but it was soon called the Sharpe ratio in articles that mentioned it. It is defined by

$$S_P = \frac{\mathrm{E}(R_P) - R_\mathrm{F}}{\sigma(R_P)}$$

where

$\mathrm{E}(R_P)$ denotes the expected return of the portfolio;
R_F denotes the return on the risk-free asset; and
$\sigma(R_P)$ denotes the standard deviation of the portfolio returns.

This ratio measures the excess return, or risk premium, of a portfolio compared with the risk-free rate, compared, this time, with the total risk of the portfolio, measured by its standard deviation. It is drawn from the capital market line. The equation of this line, which was presented at the beginning of the chapter, can be written as follows:

$$\frac{\mathrm{E}(R_P) - R_\mathrm{F}}{\sigma(R_P)} = \frac{\mathrm{E}(R_M) - R_\mathrm{F}}{\sigma(R_M)}$$

This relationship indicates that, at equilibrium, the Sharpe ratio of the portfolio to be evaluated and the Sharpe ratio of the market portfolio are equal. The Sharpe ratio actually corresponds to the slope of the market line. If the portfolio is well diversified, then its Sharpe ratio will be close to that of the market. By comparing the Sharpe ratio of the managed portfolio and the Sharpe ratio of the market portfolio, the manager can check whether the expected return on the portfolio is sufficient to compensate for the additional share of total risk that he is taking.

Since this measure is based on the total risk, it enables the relative performance of portfolios that are not very diversified to be evaluated, because the unsystematic risk taken by the manager is included in this measure. This measure is also suitable for evaluating the performance of a portfolio that represents an individual's total investment.

The Sharpe ratio is widely used by investment firms for measuring portfolio performance. The index is drawn from portfolio theory, and not the CAPM like the Treynor and Jensen indices. It does not refer to a market index and is not therefore subject to Roll's criticism.

This ratio has also been subject to generalisations since it was initially defined. It thus offers significant possibilities for evaluating portfolio performance, while remaining simple to calculate. Sharpe (1994) sums up the variations on this measure. One of the most common involves replacing the risk-free asset with a benchmark portfolio. The measure is then called the information ratio. We will describe it in more detail later in the chapter.

4.2.3 The Jensen measure

Jensen's alpha (Jensen, 1968) is defined as the differential between the return on the portfolio in excess of the risk-free rate and the return explained by the market model, or

$$E(R_P) - R_F = \alpha_P + \beta_P(E(R_M) - R_F)$$

It is calculated by carrying out the following regression:

$$R_{Pt} - R_{Ft} = \alpha_P + \beta_P(R_{Mt} - R_{Ft}) + \varepsilon_{Pt}$$

The Jensen measure is based on the CAPM. The term $\beta_P(E(R_M) - R_F)$ measures the return on the portfolio forecast by the model. α_P measures the share of additional return that is due to the manager's choices.

In order to evaluate the statistical significance of alpha, we calculate the t-statistic of the regression, which is equal to the estimated value of the alpha divided by its standard deviation. This value is obtained from the results of the regression. If the alpha values are assumed to be normally distributed, then a t-statistic greater than 2 indicates that the probability of having obtained the result through luck, and not through skill, is strictly less than 5%. In this case, the average value of alpha is significantly different from zero.

Unlike the Sharpe and Treynor measures, the Jensen measure contains the benchmark. As for the Treynor measure, only the systematic risk is taken into account. This third method, unlike the first two, does not allow portfolios with different levels of risk to be compared. The value of alpha is actually proportional to the level of risk taken, measured by the beta. To compare portfolios with different levels of risk, we can calculate the Black–Treynor ratio[8] defined by

$$\frac{\alpha_P}{\beta_P}$$

The Jensen alpha can be used to rank portfolios within peer groups. Peer groups were presented in Chapter 2. They group together portfolios that are managed in a similar manner, and that therefore have comparable levels of risk.

The Jensen measure is subject to the same criticism as the Treynor measure: the result depends on the choice of reference index. In addition, when managers practise a market timing strategy, which involves varying the beta according to anticipated movements in the market, the Jensen alpha often becomes negative, and does not then reflect the real performance of the manager. In what follows we present methods that allow this problem to be corrected by taking variations in beta into account.

4.2.4 Relationships between the different indicators and use of the indicators

It is possible to formulate the relationships between the Treynor, Sharpe and Jensen indicators.

4.2.4.1 Treynor and Jensen

If we take the equation defining the Jensen alpha, or

$$E(R_P) - R_F = \alpha_P + \beta_P(E(R_M) - R_F) \tag{4.2}$$

[8] This ratio is defined in Salvati (1997). See also Treynor and Black (1973).

and we divide on each side by β_P, then we obtain the following:

$$\frac{\mathrm{E}(R_P) - R_\mathrm{F}}{\beta_P} = \frac{\alpha_P}{\beta_P} + (\mathrm{E}(R_M) - R_\mathrm{F})$$

We then recognise the Treynor indicator on the left-hand side of the equation. The Jensen indicator and the Treynor indicator are therefore linked by the following exact linear relationship:

$$T_P = \frac{\alpha_P}{\beta_P} + (\mathrm{E}(R_M) - R_\mathrm{F})$$

4.2.4.2 Sharpe and Jensen

It is also possible to establish a relationship between the Sharpe indicator and the Jensen indicator, but this time using an approximation. To do that we replace beta with its definition, or

$$\beta_P = \frac{\rho_{PM}\sigma_P\sigma_M}{\sigma_M^2}$$

where ρ_{PM} denotes the correlation coefficient between the return on the portfolio and the return on the market index.

If the portfolio is well diversified, then the correlation coefficient ρ_{PM} is very close to 1. By replacing β_P with its approximate expression in equation (4.2) and simplifying, we obtain:

$$\mathrm{E}(R_P) - R_\mathrm{F} \approx \alpha_P + \frac{\sigma_P}{\sigma_M}(\mathrm{E}(R_M) - R_\mathrm{F})$$

By dividing each side by σ_P, we finally obtain:

$$\frac{\mathrm{E}(R_P) - R_\mathrm{F}}{\sigma_P} \approx \frac{\alpha_P}{\sigma_P} + \frac{(\mathrm{E}(R_M) - R_\mathrm{F})}{\sigma_M}$$

The portfolio's Sharpe indicator appears on the left-hand side, so

$$S_P \approx \frac{\alpha_P}{\sigma_P} + \frac{(\mathrm{E}(R_M) - R_\mathrm{F})}{\sigma_M}$$

4.2.4.3 Treynor and Sharpe

The formulas for these two indicators are very similar. If we consider the case of a well-diversified portfolio again, we can still use the following approximation for beta:

$$\beta_P \approx \frac{\sigma_P}{\sigma_M}$$

The Treynor indicator is then written as follows:

$$T_P \approx \frac{\mathrm{E}(R_P) - R_F}{\sigma_P}\sigma_M$$

Hence

$$S_P \approx \frac{T_P}{\sigma_M}$$

Table 4.1 Characteristics of the Sharpe, Treynor and Jenson indicators

Name	Risk used	Source	Criticised by Roll	Usage
Sharpe	Total (sigma)	Portfolio theory	No	Ranking portfolios with different levels of risk Not very well-diversified portfolios Portfolios that constitute an individual's total personal wealth
Treynor	Systematic (beta)	CAPM	Yes	Ranking portfolios with different levels of risk Well-diversified portfolios Portfolios that constitute part of an individual's personal wealth
Jensen	Systematic (beta)	CAPM	Yes	Ranking portfolios with the same beta

It should be noted that only the relationship between the Treynor indicator and the Jensen indicator is exact. The other two are approximations that are only valid for a well-diversified portfolio.

4.2.4.4 Using the different measures

The three indicators allow us to rank portfolios for a given period. The higher the value of the indicator, the more interesting the investment. The Sharpe ratio and the Treynor ratio are based on the same principle, but use a different definition of risk. The Sharpe ratio can be used for all portfolios. The use of the Treynor ratio must be limited to well-diversified portfolios. The Jensen measure is limited to the relative study of portfolios with the same beta.

In this group of indicators the Sharpe ratio is the one that is most widely used and has the simplest interpretation: the additional return obtained is compared with a risk indicator taking into account the additional risk taken to obtain it.

These indicators are more particularly related to equity portfolios. They are calculated by using the return on the portfolio calculated for the desired period. The return on the market is approximated by the return on a representative index for the same period. The beta of the portfolio is calculated as a linear combination of the betas of the assets that make up the portfolio, with these being calculated in relation to a reference index over the study period. The value of the indicators depends on the calculation period and performance results obtained in the past are no guarantee of future performance. Sharpe wrote that the Sharpe ratio gave a better evaluation of the past and the Treynor ratio was more suitable for anticipating future performance. Table 4.1 summarises the characteristics of the three indicators.

4.2.5 Extensions to the Jensen measure

Elton and Gruber (1995) present an additional portfolio performance measurement indicator. The principle used is the same as that of the Jensen measure, namely measuring the differential between the managed portfolio and a theoretical reference portfolio. However, the risk considered is now the total risk and the reference portfolio is no longer a portfolio located on the security market line, but a portfolio on the capital market line, with the same total risk as the portfolio to be evaluated.

More specifically, this involves evaluating a manager who has to construct a portfolio with a total risk of σ_P. He can obtain this level of risk by splitting the investment between the market portfolio and the risk-free asset. Let A be the portfolio thereby obtained. This portfolio is situated on the capital market line. Its return and risk respect the following relationship:

$$\mathrm{E}(R_A) = R_\mathrm{F} + \left(\frac{\mathrm{E}(R_M) - R_\mathrm{F}}{\sigma_M}\right)\sigma_P$$

since $\sigma_A = \sigma_P$. This portfolio is the reference portfolio.

If the manager thinks that he possesses particular stock picking skills, he can attempt to construct a portfolio with a higher return for the fixed level of risk. Let P be his portfolio. The share of performance that results from the manager's choices is then given by

$$\mathrm{E}(R_P) - \mathrm{E}(R_A) = \mathrm{E}(R_P) - R_\mathrm{F} - \left(\frac{\mathrm{E}(R_M) - R_\mathrm{F}}{\sigma_M}\right)\sigma_P$$

The return differential between portfolio P and portfolio A measures the manager's stock picking skills. The result can be negative if the manager does not obtain the expected result.

The idea of measuring managers' selectivity can be found in the Fama decomposition, which will be presented in Chapter 7. But Fama compares the performance of the portfolio with portfolios situated on the security market line, i.e. portfolios that respect the CAPM relationship.

The Jensen measure has been the object of a certain number of generalisations, which enable the management strategy used to be included in the evaluation of the manager's value-added. Among these extensions are the models that enable a market timing strategy to be evaluated. These will be developed in Section 4.3, where we will also discuss multi-factor models. The latter involve using a more precise benchmark, and will be handled in Chapter 6.

Finally, the modified versions of the CAPM, presented at the end of Section 4.1, can be used instead of the traditional CAPM to calculate the Jensen alpha. The principle remains the same: the share of the return that is not explained by the model gives the value of the Jensen alpha.

With the Black model, the alpha is characterised by

$$\mathrm{E}(R_P) - \mathrm{E}(R_Z) = \alpha_P + \beta_P(\mathrm{E}(R_M) - \mathrm{E}(R_Z))$$

With the Brennan model, the alpha is characterised by

$$\mathrm{E}(R_P) - R_\mathrm{F} = \alpha_P + \beta_P(\mathrm{E}(R_M) - R_\mathrm{F} - T(D_M - R_\mathrm{F})) + T(D_P - R_\mathrm{F})$$

where D_P is equal to the weighted sum of the dividend yields of the assets in the portfolio, or

$$D_P = \sum_{i=1}^{n} x_i D_i$$

x_i denotes the weight of asset i in the portfolio. The other notations are those that were used earlier.

We can go through all the models cited in this way. For each case, the value of α_P is estimated through regression.

4.2.6 The tracking-error

The tracking-error is a risk indicator that is used in the analysis of benchmarked funds. Benchmarked management involves constructing portfolios with the same level of risk as an index, or a portfolio chosen as a benchmark, while giving the manager the chance to deviate from the benchmark composition, with the aim of obtaining a higher return. This assumes that the manager possesses particular stock picking skills. The tracking-error then allows the risk differentials between the managed portfolio and the benchmark portfolio to be measured. It is defined by the standard deviation of the difference in return between the portfolio and the benchmark it is replicating, or

$$TE = \sigma(R_P - R_B)$$

where R_B denotes the return on the benchmark portfolio.

The lower the value, the closer the risk of the portfolio to the risk of the benchmark. Benchmarked management requires the tracking-error to remain below a certain threshold, which is fixed in advance. To respect this constraint, the portfolio must be reallocated regularly as the market evolves. It is necessary however to find the right balance between the frequency of the reallocations and the transaction costs that they incur, which have a negative impact on portfolio performance. The additional return obtained, measured by alpha, must also be sufficient to make up for the additional risk taken on by the portfolio. To check this, we use another indicator: the information ratio.

4.2.7 The information ratio

The information ratio, which is sometimes called the appraisal ratio, is defined by the residual return of the portfolio compared with its residual risk. The residual return of a portfolio corresponds to the share of the return that is not explained by the benchmark. It results from the choices made by the manager to overweight securities that he hopes will have a return greater than that of the benchmark. The residual, or diversifiable, risk measures the residual return variations. Sharpe (1994) presents the information ratio as a generalisation of his ratio, in which the risk-free asset is replaced by a benchmark portfolio. The information ratio is defined through the following relationship:

$$IR = \frac{E(R_P) - E(R_B)}{\sigma(R_P - R_B)}$$

We recognise the tracking-error in the denominator. The ratio can also be written as follows:

$$IR = \frac{\alpha_P}{\sigma(e_P)}$$

where α_P denotes the residual portfolio return, as defined by Jensen, and $\sigma(e_P)$ denotes the standard deviation of this residual return.

As specified above, this ratio is used in the area of benchmarked management. It allows us to check that the risk taken by the manager, in deviating from the benchmark, is sufficiently rewarded. It constitutes a criterion for evaluating the manager. Managers seek to maximise its value, i.e. to reconcile a high residual return and a low tracking-error. It is important to look at the value of the information ratio and the value of the tracking-error together. For the same information ratio value, the lower the tracking-error the higher the chance that the manager's performance will persist over time.

The information ratio is therefore an indicator that allows us to evaluate the manager's level of information compared with the public information available, together with his skill in achieving a performance that is better than that of the average manager. Since this ratio does not take the systematic portfolio risk into account, it is not appropriate for comparing the performance of a well-diversified portfolio with that of a portfolio with a low degree of diversification.

The information ratio also allows us to estimate a suitable number of years for observing the performance, in order to obtain a certain confidence level for the result. To do so, we note that there is a link between the t-statistic of the regression, which provides the alpha value, and the information ratio. The t-statistic is equal to the quotient of alpha and its standard deviation, and the information ratio is equal to the same quotient, but this time using annualised values. We therefore have

$$IR \approx \frac{t_{\text{stat}}}{\sqrt{T}}$$

where T denotes the length of the period, expressed in years, during which we observed the returns. The number of years required for the result obtained to be significant, with a given level of probability, is therefore calculated by the following relationship:

$$T = \left[\frac{t_{\text{stat}}}{IR}\right]^2$$

For example, a manager who obtains an average alpha of 2.5% with a tracking-error of 4% has an information ratio equal to 0.625. If we wish the result to be significant to 95%, then the value of the t-statistic is 1.96, according to the normal distribution table, and the number of years it is necessary to observe the portfolio returns is

$$T = \left[\frac{1.96}{0.625}\right]^2 = 9.8 \text{ years}$$

This shows clearly that the results must persist over a long period to be truly significant. We should note, however, that the higher the manager's information ratio, the more the number of years decreases. The number of years also decreases if we consider a lower level of probability, by going down, for example, to 80%.

The calculation of the information ratio has been presented by assuming that the residual return came from the Jensen model. More generally, this return can come from a multi-index or multi-factor model. We will discuss these models in Chapter 6.

4.2.8 The Sortino ratio

An indicator such as the Sharpe ratio, based on the standard deviation, does not allow us to know whether the differentials compared with the mean were produced above or below the mean.

In Chapter 2 we introduced the notion of semi-variance and its more general versions. This notion can then be used to calculate the risk-adjusted return indicators that are more specifically appropriate for asymmetrical return distributions. This allows us to evaluate the portfolios obtained through an optimisation algorithm using the semi-variance instead of the variance. The best known indicator is the Sortino ratio (cf. Sortino and Price, 1994). It is defined on the same principle as the Sharpe ratio. However, the risk-free rate is replaced with the minimum acceptable return (*MAR*), i.e. the return below which the investor does not wish to drop, and the standard deviation of the returns is replaced with the standard deviation of the returns that are below the *MAR*, or

$$\text{Sortino ratio} = \frac{E(R_P) - MAR}{\sqrt{\dfrac{1}{T} \sum_{\substack{t=0 \\ R_{Pt} < MAR}}^{T} (R_{Pt} - MAR)^2}}$$

4.2.9 Recently developed risk-adjusted return measures

Specialised firms that study investment fund performance develop variations on the traditional measures, essentially on the Sharpe ratio. These measures are used to rank the funds and attribute management quality labels. We can cite, for example, Morningstar's rankings.

4.2.9.1 The Morningstar rating system[9]

The Morningstar measure, which is called a risk-adjusted rating (*RAR*), is very widely used in the United States. This ranking system was first developed in 1985 by the firm Morningstar. In July 2002, Morningstar introduced some modifications to improve its methodology. The measure differs significantly from more traditional measures such as the Sharpe ratio and its different forms. The evaluation of funds is based on a system of stars. Sharpe (1998) presents the method used by Morningstar and describes its properties. He compares it with other types of measure and describes the limitations of the ranking system.

The principle of the Morningstar measure is to rank different funds that belong to the same peer group. The *RAR* for a fund is calculated as the difference between its relative return and its relative risk, or

$$RAR_{P_i} = RR_{P_i} - RRisk_{P_i}$$

where RR_{P_i} denotes the relative return for fund P_i; and $RRisk_{P_i}$ denotes the relative risk for fund P_i.

The relative return and the relative risk for the fund are obtained by dividing, respectively, the return and the risk of the fund by a quantity, called the base, which is common to all the funds in the peer group, or

$$RR_{P_i} = \frac{R_{P_i}}{BR_g}$$

and

$$RRisk_{P_i} = \frac{Risk_{P_i}}{BRisk_g}$$

where g denotes the peer group containing the fund P_i;

[9] Cf. Melnikoff (1998) and see Sharpe's web site (http://www.stanford.edu/~wfsharpe/home.htm) for a series of articles describing the calculation methods.

R_{P_i} denotes the return on fund P_i, in excess of the risk-free rate;
$Risk_{P_i}$ denotes the risk of fund P_i;
BR_g denotes the base used to calculate the relative returns of all the funds in the group;
$BRisk_g$ denotes the base used to calculate the relative risks of all the funds in the group.

In the first version of the methodology, the risk of a fund was measured by calculating the average of the negative values of the fund's monthly returns in excess of the short-term risk-free rate and by taking the opposite sign to obtain a positive quantity:

$$Risk_{P_i} = -\frac{1}{T}\sum_{t=1}^{T}\min\left(R_{P_it}, 0\right)$$

where T denotes the number of months in the period being studied; and R_{P_it} denotes the monthly return of fund P_i, in excess of the risk-free rate.

Risk calculation has been modified in the new version of the star rating. Risk is measured by monthly variations in fund returns and now takes not only downside risk but also upside volatility into account, but with more emphasis on downward volatility. Funds with highly volatile returns are penalised, whether the volatility is upside or downside. The advantages of this improvement can be understood by looking at Internet funds. These funds were not considered risky in 1999, as they only exhibited upside volatility. But their extreme gains indicated a serious potential for extreme losses, as has been demonstrated since. The new risk measure would have attributed a higher level of risk to those funds than the previous measure did. As a result, the possibility of strong short-term performance masking the inherent risk of a fund has now been reduced and it is more difficult for high-risk funds to earn high star ratings.

The base that is used to calculate the relative return of the funds is obtained by calculating the average return of the funds in the group. If the value obtained is greater than the risk-free rate for the period, then we use the result obtained, otherwise we use the value of the risk-free rate. We therefore have

$$BR_g = \max\left(\frac{1}{n}\sum_{i=1}^{n}R_{P_i}, R_F\right)$$

where n denotes the number of funds contained in the peer group; and R_F denotes the risk-free rate.

The base used to calculate the relative risk is obtained by calculating the average of the risks of the funds in the peer group, or

$$BRisk_g = \frac{1}{n}\sum_{i=1}^{n}Risk_{P_i}$$

In 1985, Morningstar defined four peer groups to establish its rankings: domestic stock funds, international stock funds, taxable bond funds and tax-exempt municipal bond funds. However, these four categories appear to be too few to make truly adequate comparisons. The improved star rating methodology[10] now uses 48 specific equity and debt peer groups. For example, equity funds are classified according to their capitalisation (large-cap, mid-cap and small-cap) and whether they are growth, value or blend. International stock funds are now subdivided into different parts of the world. By only comparing funds with funds from the

[10] For more details, see Morningstar's web site www.morningstar.com, from which it is possible to visit the specific web sites for each country.

same well-defined category, those that are providing superior risk-adjusted performance will be more accurately identified. For example, during periods favourable to large-cap stocks, large-cap funds received a high percentage of five-star rankings when evaluated in the broad domestic equity group. With the new system, only the best funds will receive five stars, as large-cap funds will only be compared with large-cap funds.

The ranking is then produced as follows. Each fund is attached to a single peer group. The funds in a peer group are ranked in descending order of their *RAR*. A number of stars is then attributed to each fund according to its position in the distribution of *RAR* values. The funds in the top 10% of the distribution obtain five stars; those in the following 22.5% obtain four stars; those in the following 35% obtain three stars; those in the next 22.5% obtain two stars; and, finally, those in the bottom 10% obtain one star.

The Morningstar measure is based on an investment period of one month, although funds are in fact held for longer periods, and a decrease in one month can be compensated for by an increase in the following month. This measure is not therefore very appropriate for measuring the risk of funds that are held over a long period.

4.2.9.2 Actuarial approach

In this approach (see Melnikoff, 1998) the investor's aversion to risk is characterised by a constant, W, which measures his gain–shortfall equilibrium, i.e. the relationship between the expected gain desired by the investor to make up for a fixed shortfall risk. The average annual risk-adjusted return is then given by

$$RAR = \dot{R} - (W - 1)S$$

where

S denotes the average annual shortfall rate;
W denotes the weight of the gain–shortfall aversion; and
R denotes the average annual rate of return obtained by taking all the observed returns.

For an average individual, W is equal to two, which means that the individual will agree to invest if the expected amount of his gain is double the shortfall. In this case, we have simply

$$RAR = R - S$$

4.2.9.3 Analysis based on the VaR

The VaR was defined in Chapter 2 and the different methods for calculating it were briefly presented. As a reminder, the VaR measures the risk of a portfolio as the maximum amount of loss that the portfolio can sustain for a given level of confidence. We may then wish to use this definition of risk to calculate a risk-adjusted return indicator to evaluate the performance of a portfolio. In order to define a logical indicator, we divide the VaR by the initial value of the portfolio and thus obtain a percentage loss compared with the total value of the portfolio. We then calculate a Sharpe-like type of indicator in which the standard deviation is replaced with a risk indicator based on the VaR, or

$$\frac{R_P - R_F}{\dfrac{VaR_P}{V_P^0}}$$

where

R_P denotes the return on the portfolio;
R_F denotes the return on the risk-free asset;
VaR_P denotes the VaR of the portfolio;
V_P^0 denotes the initial value of the portfolio.

This type of ratio can only be compared for different portfolios if the portfolios' VaR has been evaluated for the same confidence threshold.

Furthermore, Dowd (1999) proposes an approach based on the VaR to evaluate an investment decision. We consider the case of an investor who holds a portfolio that he is thinking of modifying, by introducing, for example, a new asset. He will study the risk and return possibilities linked to a modification of the portfolio and choose the situation for which the risk-return balance seems to be sufficiently favourable. To do that, he could decide to define the risk in terms of the increase in the portfolio's VaR. He will change the portfolio if the resulting incremental VaR (IVaR) is sufficiently low compared with the return that he can expect. This can be formalised as a decision rule based on Sharpe's decision rule.

Sharpe's rule states that the most interesting asset in a set of assets is the one that has the highest Sharpe ratio. By calculating the existing Sharpe ratio and the Sharpe ratio for the modified portfolio and comparing the results, we can then judge whether the planned modification of the portfolio is desirable.

By using the definition of the Sharpe ratio, we find that it is useful to modify the portfolio if the returns and standard deviations of the portfolio before and after the modification are linked by the following relationship:

$$\frac{R_P^{\text{new}}}{\sigma_{R_P^{\text{new}}}} \geq \frac{R_P^{\text{old}}}{\sigma_{R_P^{\text{old}}}}$$

where R_P^{old} and R_P^{new} denote, respectively, the return on the portfolio before and after the modification; and $\sigma_{R_P^{\text{old}}}$ and $\sigma_{R_P^{\text{new}}}$ denote, respectively, the standard deviation of the portfolio before and after the modification.

We assume that part of the new portfolio is made up of the existing portfolio, in proportion $(1-a)$, and the other part is made up of asset A in proportion a.

The return on this portfolio is written as follows:

$$R_P^{\text{new}} = aR_A + (1-a)R_P^{\text{old}}$$

where R_A denotes the return on asset A.

By replacing R_P^{new} with its expression in the inequality between the Sharpe ratios, we obtain:

$$\frac{aR_A + (1-a)R_P^{\text{old}}}{\sigma_{R_P^{\text{new}}}} \geq \frac{R_P^{\text{old}}}{\sigma_{R_P^{\text{old}}}}$$

which finally gives

$$R_A \geq R_P^{\text{old}} + \frac{R_P^{\text{old}}}{a}\left(\frac{\sigma_{R_P^{\text{new}}}}{\sigma_{R_P^{\text{old}}}} - 1\right)$$

This relationship indicates the inequality that the return on asset A must respect for it to be advantageous to introduce it into the portfolio. The relationship depends on proportion a. It shows that the return on asset A must be at least equal to the return on the portfolio before the modification, to which is added a factor that depends on the risk associated with the acquisition of asset A. The higher the risk, the higher the adjustment factor and the higher the return on asset A will have to be.

Under certain assumptions, this relationship can be expressed through the VaR instead of the standard deviation. If the portfolio returns are normally distributed, then the VaR of the portfolio is proportional to its standard deviation, or

$$VaR = -\alpha\sigma_{R_P} W$$

where

α denotes the confidence parameter for which the VaR is estimated;
W is a parameter that represents the size of the portfolio; and
σ_{R_P} is the standard deviation of the portfolio returns.

By using this expression of the VaR, we can calculate

$$\frac{VaR^{\text{new}}}{VaR^{\text{old}}} = \frac{W^{\text{new}}\sigma_{R_P^{\text{new}}}}{W^{\text{old}}\sigma_{R_P^{\text{old}}}}$$

which enables us to obtain the following relationship:

$$\frac{\sigma_{R_P^{\text{new}}}}{\sigma_{R_P^{\text{old}}}} = \frac{VaR^{\text{new}}}{VaR^{\text{old}}}\frac{W^{\text{old}}}{W^{\text{new}}}$$

We assume that the size of the portfolio is conserved. We therefore have $W^{\text{old}} = W^{\text{new}}$.

We therefore obtain simply, after substituting into the return on A relationship:

$$R_A \geq R_P^{\text{old}} + \frac{R_P^{\text{old}}}{a}\left(\frac{VaR^{\text{new}}}{VaR^{\text{old}}} - 1\right)$$

The incremental VaR between the new portfolio and the old portfolio, denoted by IVaR, is equal to the difference between the old and new value, or $IVaR = VaR^{\text{new}} - VaR^{\text{old}}$.

By replacing in the inequality according to the IVaR, we obtain:

$$R_A \geq R_P^{\text{old}} + \frac{R_P^{\text{old}}}{a}\left(\frac{IVaR}{VaR^{\text{old}}}\right) = R_P^{\text{old}}\left(1 + \frac{1}{a}\frac{IVaR}{VaR^{\text{old}}}\right)$$

By defining the function η_A as

$$\eta_A(VaR) = \frac{1}{a}\frac{IVaR}{VaR^{\text{old}}}$$

we can write

$$R_A \geq (1 + \eta_A(VaR))R_P^{\text{old}}$$

where $\eta_A(VaR)$ denotes the percentage increase in the VaR occasioned by the acquisition of asset A, divided by the proportion invested in asset A.

4.2.9.4 Measure taking the management style into account

The risk-adjusted performance measures enable a fund to be evaluated in comparison with the market portfolio, but do not take the manager's investment style into account. The style, however, may be imposed by the management mandate constraints rather than chosen by the manager. In this case it is more useful to compare management results with a benchmark that accurately represents the manager's style, rather than comparing them with a broad benchmark representing the market (cf. Lobosco, 1999). The idea of using tailored benchmarks that are adapted to the manager's investment style comes from the work of Sharpe (1992). We have already mentioned these benchmarks in the section devoted to benchmarks in Chapter 2, and we will come back to them in Chapter 6 with multi-factor models.

Lobosco (1999) proposes a measure called *SRAP* (Style/Risk-Adjusted Performance). This is a risk-adjusted performance measure that includes the management style as defined by Sharpe. It was inspired by the work of Modigliani and Modigliani (1997), who defined an equation that enabled the annualised risk-adjusted performance (*RAP*) of a fund to be measured in relation to the market benchmark, or

$$RAP_P = \frac{\sigma_M}{\sigma_P}(R_P - R_F) + R_F$$

where

σ_M denotes the annualised standard deviation of the market returns;
σ_P denotes the annualised standard deviation of the returns of fund P;
R_P denotes the annualised return of fund P; and
R_F denotes the risk-free rate.

This relationship is drawn directly from the capital market line. If we were at equilibrium, we would have $RAP_P = R_M$, where R_M denotes the annualised average market return.

The relationship therefore allows us to look at the performance of the fund in relation to that of the market. The most interesting funds are those with the highest *RAP* value. To obtain a relative measure, one just calculates the difference between the *RAP* for the fund and the *RAP* for the benchmark, with the benchmark's *RAP* measure being simply equal to its return.

The first step in measuring the performance of a fund, when taking the investment style into account, is to identify the combination of indices that best represents the manager's style. We then calculate the differential between the fund's *RAP* measure and the *RAP* measure of its Sharpe benchmark.

Lobosco gives the example of a fund with an annualised performance of −1.72% and a standard deviation of 17.48%. The market portfolio is represented by the Russell 3000 index, the performance of which for the same period is 16.54% with a standard deviation of 11.52%. The risk-free rate is 5.21%.

The risk-adjusted performance of this fund is therefore

$$RAP(Fund) = \frac{11.52}{17.48}(-1.72 - 5.21) + 5.21 = 0.64\%$$

Its performance in relation to the market portfolio is

$$RelativeRAP = RAP(Fund) - RAP(Market) = 0.64 - 16.54 = -15.90\%$$

If we now observe that the style of this fund corresponds to a benchmark, 61% of which is made up of the Russell 2000 index of growth stocks and 39% of the Russell 2000 index of growth stocks, the performance of this benchmark is now 2.73% with a standard deviation of 13.44%.

The risk-adjusted performance of this benchmark is given by

$$RAP(SharpeBenchmark) = \frac{11.52}{13.44}(2.73 - 5.21) + 5.21 = 3.08\%$$

and the relative performance of the portfolio compared to this benchmark is given by

$$RelativeRAP = RAP(Fund) - RAP(SharpeBenchmark) = 0.64 - 3.08 = -2.44\%$$

The relative performance of the fund is again negative, but the differential is much lower than compared with the whole market. The management style-adjusted performance measure is therefore a useful additional measure.

4.2.9.5 Risk-adjusted performance measure in the area of multimanagement

Muralidhar (2001) has developed a new risk-adjusted performance measure that allows us to compare the performance of different managers within a group of funds with the same objectives (a peer group). This measure can be grouped with the existing information ratio, the Sharpe ratio and the Modigliani and Modigliani measure, but it does contribute new elements. It includes not only the standard deviations of each portfolio, but also the correlation of each portfolio with the benchmark and the correlations between the portfolios themselves. The method proposed by Muralidhar allows us to construct portfolios that are split optimally between a risk-free asset, a benchmark and several managers, while taking the investors' objectives into account, both in terms of risk and, above all, the relative risk compared with the benchmark.

The principle involves reducing the portfolios to those with the same risk in order to be able to compare their performance. This is the same idea as in Modigliani and Modigliani (1997) who compared the performance of a portfolio and its benchmark by defining transformations in such a way that the transformed portfolio and benchmark had the same standard deviation.

To create a correlation-adjusted performance measure, Muralidhar considers an investor who splits his portfolio between a risk-free asset, a benchmark and an investment fund. We assume that this investor accepts a certain level of annualised tracking-error compared with his benchmark, which we call the objective tracking-error. The investor wishes to obtain the highest risk-adjusted value of alpha for a given portfolio tracking-error and variance. We define as a, b and $(1 - a - b)$ the proportions invested respectively in the investment fund, the benchmark B and the risk-free asset F. The portfolio thereby obtained is said to be correlation-adjusted. It is denoted by the initials CAP (for correlation-adjusted portfolio). The return on this portfolio is given by

$$R(CAP) = aR(manager) + bR(B) + (1 - a - b)R(F)$$

The proportions to be held must be chosen in an appropriate manner so that the portfolio obtained has a tracking-error equal to the objective tracking-error and its standard deviation is equal to the standard deviation of the benchmark.

The search for the best return, in view of the constraints, leads to the calculation of optimal proportions that depend on the standard deviations and correlations of the different elements in the portfolio. The problem is considered here with a single fund, but it can be generalised to the case of several funds, to handle the case of portfolios split between several managers, and to find the optimal allocation between the different managers. The formulas that give the optimal weightings in the case of several managers have the same structure as those obtained in the case of a single manager, but they use the weightings attributed to each manager together with the correlations between the managers.

Once the optimal proportions have been calculated, the return on the CAP has been determined entirely. By carrying out the calculation for each fund being studied, we can rank the different funds.

The Muralidhar measure is certainly useful compared with the risk-adjusted performance measure that had been developed previously. We observe that the Sharpe ratio, the information ratio and the Modigliani and Modigliani measure turn out to be insufficient to allow investors to rank different funds and to construct their optimal portfolio. These risk-adjusted measures only include the standard deviations of the portfolios and the benchmark, even though it is also necessary to include the correlations between the portfolios and between the portfolios and the benchmark. The Muralidhar model therefore provides a more appropriate risk-adjusted performance measure because it takes into account both the differences in standard deviation and the differences in correlations between the portfolios. We see that it produces a ranking of funds that is different from that obtained with the other measures. In addition, neither the information ratio nor the Sharpe ratio indicates how to construct portfolios in order to produce the objective tracking-error, while the Muralidhar measure provides the composition of the portfolios that satisfy the investors' objectives.

The composition of the portfolio obtained through the Muralidhar method enables us to solve the problem of an institutional investor's optimal allocation between active and passive management, with the possible use of a leverage effect to improve the risk-adjusted performance.

All the measures described in this section enable different investment funds to be ranked based on past performance. The calculations can be carried out over several successive periods on the basis that the more stable the ranking, the easier it will be to anticipate consistent results in the future.

4.3 EVALUATING THE MANAGEMENT STRATEGY WITH THE HELP OF MODELS DERIVED FROM THE CAPM: TIMING ANALYSIS

The first performance measurement indicators, which were drawn from portfolio theory and the CAPM (Sharpe, Treynor and Jensen), assume that portfolio risk is stationary. They measure the additional return obtained, compared with the level of risk taken, by considering the average value of the risk over the evaluation period. As a result, the measures only take the stock picking aspect into account. However, there is an investment management strategy, namely market timing, that involves modifying the level of the portfolio's exposure to market risk, measured by its beta, according to its anticipated evolution. To evaluate this type of strategy, one must turn to other models.

In this section we first present two performance analysis models, again based on the CAPM, which enable variations in the portfolio's beta over the investment period to be taken into account. They actually involve statistical tests, which allow for qualitative evaluation of a market timing strategy, when that strategy is followed for the portfolio. These models allow us to measure the portfolio's Jensen alpha, and to assess whether the result was obtained through the right investment decisions being taken at the right time or through luck. This section also presents a decomposition of the Jensen measure, which enables timing to be evaluated. The methods for implementing the market timing strategy itself will be presented in Chapter 7, which is devoted to the description and quantitative evaluation of the investment process.

4.3.1 The Treynor and Mazuy (1966) method[11]

This model is a quadratic version of the CAPM, which provides us with a better framework for taking into account the adjustments made to the portfolio's beta, and thus for evaluating a manager's market timing capacity. A manager who anticipates market evolutions correctly will lower his portfolio's beta when the market falls. His portfolio will thus depreciate less than if he had not made the adjustment. Similarly, when he anticipates a rise in the market, he increases his portfolio's beta, which enables him to make higher profits. The relationship between the portfolio return and the market return, in excess of the risk-free rate, should therefore be better approximated by a curve than by a straight line. The model is formulated as follows:

$$R_{Pt} - R_{Ft} = \alpha_P + \beta_P(R_{Mt} - R_{Ft}) + \delta_P(R_{Mt} - R_{Ft})^2 + \varepsilon_{Pt}$$

where

R_{Pt} denotes the portfolio return vector for the period studied;
R_{Mt} denotes the vector of the market returns for the same period, measured with the same frequency as the portfolio returns; and
R_{Ft} denotes the rate of the risk-free asset over the same period.

The α_P, β_P and δ_P coefficients in the equation are estimated through regression. If δ_P is positive and significantly different from zero, then we can conclude that the manager has successfully practised a market timing strategy.

This model was formulated empirically by Treynor and Mazuy (1966). It was then theoretically validated by Jensen (1972b) and Bhattacharya and Pfleiderer (1983).

4.3.2 The Henriksson and Merton (1981) and Henriksson (1984) models[12]

There are in fact two models: a non-parametric model and a parametric model. They are based on the same principle, but the parametric model seems to be more natural to implement. The non-parametric model is less frequently mentioned in the literature: we find it in Farrell (1997) and in Philips *et al.* (1996).

The non-parametric version of the model is older, and does not use the CAPM. It was developed by Merton (1981) and uses options theory. The principle is that of an investor who can split his portfolio between a risky asset and a risk-free asset, and who modifies the split over time according to his anticipations on the relative performance of the two assets. If the strategy is perfect, then the investor only holds stocks when their performance is better than that of the risk-free asset and only holds cash in the opposite case. The portfolio can be modelled by an investment in cash and a call on the better of the two assets. If the forecasts are not perfect, then the manager will only hold a fraction of options f, situated between −1 and 1. The value of f allows us to evaluate the manager. To do so, we define two conditional probabilities:

P_1 denotes the probability of making an accurate forecast, given that the stocks beat the risk-free asset;

[11] Cf. Broquet and van den Berg (1992), Elton and Gruber (1995), Farrell (1997), Grandin (1998), Jacquillat and Solnik (1997), Sharpe (1985), Taggart (1996), and Lhabitant (1994).
[12] Cf. Merton (1981), Henriksson and Merton (1981) and Henriksson (1984), and also Broquet and van den Berg (1992), Elton and Gruber (1995), Farrell (1997), Grandin (1998), Grinold and Kahn (1995), Jacquillat and Solnik (1997), Sharpe (1985), Taggart (1996), and Lhabitant (1994).

P_2 denotes the probability of making an accurate forecast, given that the risk-free asset beats the stocks.

We then have $f = P_1 + P_2 - 1$ and the manager has a market timing capacity if $f > 0$, i.e. if the sum of the two conditional probabilities is greater than one.

f can be estimated by using the following formula:

$$I_{t-1} = \alpha_0 + \alpha_1 y_t + \varepsilon_t$$

where $I_{t-1} = 1$ if the manager forecasts that the stocks will perform better than the risk-free asset during month t, otherwise 0; and $y_t = 1$ if the stocks actually did perform better than the risk-free asset, otherwise 0.

The coefficients in the equation are estimated through regression. α_0 gives the estimation of $1 - P_1$ and α_1 gives the estimation of $P_1 + P_2 - 1$. We then test the hypothesis $\alpha_1 > 0$.

Henriksson and Merton (1981) then developed a parametric model. The idea is still the same, but the formulation is different. It consists of a modified version of the CAPM which takes the manager's two risk objectives into account, depending on whether he forecasts that the market return will or will not be better than the risk-free asset return. The model is presented in the following form:

$$R_{Pt} - R_{Ft} = \alpha_P + \beta_{1P}(R_{Mt} - R_{Ft}) - \beta_{2P} D_t (R_{Mt} - R_{Ft}) + \varepsilon_{Pt}$$

where

$$\begin{aligned} D_t &= 0, \quad \text{if } R_{Mt} - R_{Ft} > 0 \\ D_t &= -1, \quad \text{if } R_{Mt} - R_{Ft} < 0 \end{aligned}$$

The α_P, β_{1P} and β_{2P} coefficients in the equation are estimated through regression. The β_{2P} coefficient allows us to evaluate the manager's capacity to anticipate market evolution. If β_{2P} is positive and significantly different from zero, then the manager has a good timing capacity.

These models have been presented while assuming that the portfolio was invested in stocks and cash. More generally, they are valid for a portfolio that is split between two categories of assets, with one riskier than the other, for example stocks and bonds, and for which we adjust the composition according to anticipations on their relative performance.

4.3.3 Decomposition of the Jensen measure and evaluation of timing

The Jensen measure has been subject to numerous criticisms, the main one being that a negative performance can be attributed to a manager who practices market timing. As we mentioned above, this comes from the fact that the model uses an average value for beta, which tends to overestimate the portfolio risk, while the manager varies his beta between a high beta and a low beta according to his expectations for the market. Grinblatt and Titman (1989) present a decomposition of the Jensen measure in three terms: a term measuring the bias in the beta evaluation, a timing term and a selectivity term.

In order to establish this decomposition, we assume that there are n risky assets traded on a frictionless market, i.e. no transaction costs, no taxes and no restrictions on short selling. We assume that there is a risk-free asset. The assumptions are therefore those of the CAPM. We seek to evaluate the investor's performance over T time periods, by looking at the risk-adjusted returns of his portfolio. We denote r_{it} as the return on asset i in excess of the risk-free rate for period t; and x_{it} as the weight of asset i in the investor's portfolio for period t.

The return on the investor's portfolio for period t, in excess of the risk-free rate, is then given by

$$r_{Pt} = \sum_{i=1}^{n} x_{it} r_{it}$$

We denote by r_{Bt} the return in excess of the risk-free rate of a portfolio that is mean–variance efficient from an uninformed investor's viewpoint. We can then write

$$r_{it} = \beta_i r_{Bt} + \varepsilon_{it}$$

where

$$\beta_i = \frac{\mathrm{cov}(r_{it}, r_{Bt})}{\mathrm{var}(r_{Bt})}$$

and

$$\mathrm{E}(\varepsilon_{it}) = 0$$

The portfolio return is then written as

$$r_{Pt} = \beta_{Pt} r_{Bt} + \varepsilon_{Pt}$$

where

$$\beta_{Pt} = \sum_{i=1}^{n} x_{it} \beta_i$$

and

$$\varepsilon_{Pt} = \sum_{i=1}^{n} x_{it} \varepsilon_{it}$$

In order to establish the decomposition, we consider the limit, in the probabilistic sense, of the Jensen measure, which is written as follows:

$$J_P = \hat{r}_P - b_P \hat{r}_B$$

where

b_p is the probability limit of the coefficient from the time-series regression of the portfolio returns against the reference portfolio series of returns;
$\hat{r}_P$ is the probability limit of the sample mean of the r_{Pt} series; and
$\hat{r}_B$ is the probability limit of the sample mean of the r_{Bt} series.

Formally, the probability limit of a variable is defined as

$$\hat{r}_P = p\lim\left[\frac{1}{T}\sum_{t=1}^{T} r_{Pt}\right]$$

It should be noted that b_P can be different from $\hat{\beta}_P$. This is the case when a manager practises market timing. $\hat{\beta}_P$ is then a weighted mean of the two betas used for the portfolio, while b_P is the regression coefficient obtained, without concerning oneself with the fact that the manager practises market timing.

We can write

$$\hat{r}_P = p\lim\left[\frac{1}{T}\sum_{t=1}^{T} r_{Pt}\right]$$

or, by replacing r_{Pt} with its expression:

$$\hat{r}_P = p\lim\left[\frac{1}{T}\sum_{t=1}^{T}(\beta_{Pt}r_{Bt} + \varepsilon_{Pt})\right]$$

By arranging the terms in the expression we obtain:

$$\hat{r}_P = \hat{\beta}_P\hat{r}_B + p\lim\left[\frac{1}{T}\sum_{t=1}^{T}\beta_{Pt}(r_{Bt} - \hat{r}_B)\right] + \hat{\varepsilon}_P$$

By using this formula in the Jensen measure expression we obtain:

$$J_P = (\hat{\beta}_P - b_P)\hat{r}_B + p\lim\left[\frac{1}{T}\sum_{t=1}^{T}\beta_{Pt}(r_{Bt} - \hat{r}_B)\right] + \hat{\varepsilon}_P$$

This expression reveals three distinct terms:

1. a term that results from the bias in estimated beta: $(\hat{\beta}_P - b_P)\hat{r}_B$;
2. a term that measures timing:

$$p\lim\left[\frac{1}{T}\sum_{t=1}^{T}\beta_{Pt}(r_{Bt} - \hat{r}_B)\right];$$

3. a term that measures selectivity: $\hat{\varepsilon}_P$.

If the weightings of the portfolio to be evaluated are known, then the three terms can be evaluated separately. When the manager has no particular information in terms of timing, $\hat{\beta}_P = b_P$.

4.4 MEASURING THE PERFORMANCE OF INTERNATIONALLY DIVERSIFIED PORTFOLIOS: EXTENSIONS TO THE CAPM

Modern portfolio theory has demonstrated the usefulness of diversification in reducing portfolio risk. By enlarging the universe of available securities, international investment[13] offers additional diversification possibilities. Assets from different countries often have low levels of correlation. It is therefore possible to put together less risky portfolios than by limiting oneself to a single country. However, currency risk, which was defined in Chapter 2, has to be taken into consideration. The performance of international portfolios can then be evaluated with specific models, based on an international version of the CAPM.

[13] The advantages of international diversification are detailed in Chapter 11 of Jacquillat and Solnik (1997) and Chapter 1 of Farrell (1997).

4.4.1 International Asset Pricing Model[14]

Several authors have developed international versions of the CAPM. Among these, we could mention Solnik's model (1974a, 1974b), which is called the International Asset Pricing Model (IAPM). This model was established by following a similar framework to that used to obtain the continuous time version of the CAPM in the national case. The reference portfolio is now the worldwide market portfolio. The most widely used index in the United States, as an approximation of this portfolio, is the Morgan Stanley Capital Index (MSCI) Europe, Asia and Far East (EAFE). This is an index that is weighted according to the stock market capitalisations of each country. It covers more than 2000 companies from 21 countries. This model uses a risk-free rate from the country of asset i and an average worldwide risk-free rate, obtained by making up a portfolio of risk-free assets from different countries in the world. The weightings used are again the same as those used for the worldwide market portfolio. Solnik establishes the following relationship:

$$\mathrm{E}(R_i) = R_{\mathrm{F}_i} + \beta_i\left(E\left(R_{M_M}\right) - R_{\mathrm{F}_M}\right)$$

where

β_i denotes the international systematic risk of security i, i.e. calculated in relation to the worldwide market portfolio;
R_{F_i} denotes the rate of the risk-free asset in the country of security i;
R_{F_M} denotes the rate of the average worldwide risk-free asset; and
R_{M_M} denotes the return on the worldwide market portfolio.

All the rates of return are expressed in the currency of the asset i country.

4.4.2 McDonald's model[15]

McDonald (1973) proposed a performance measure which is an extension to the Jensen measure. His model applies to a portfolio of stocks invested in the French and American markets. It is written as follows:

$$R_{Pt} - R_{\mathrm{F}t} = \Phi_P + \beta^*_{P1}(R_{M1,t} - R_{\mathrm{F}t}) + \beta^*_{P2}(R_{M2,t} - R_{\mathrm{F}t}) + e_{Pt}$$

where

$R_{M1,t}$ denotes the rate of return of the French market in period t;
$R_{M2,t}$ denotes the rate of return of the American market in period t;
$R_{\mathrm{F}t}$ denotes the rate of return of the risk-free asset in the French market in period t;
$\beta^*_{P1} = x_1\beta_{P1}$ and $\beta^*_{P2} = x_2\beta_{P2}$, with x_1 and x_2 the proportions of the fund invested in each of the two markets and β_{P1} and β_{P2} the fund's coefficients of systematic risk compared to each of the two markets.

The overall excess performance of the fund Φ_P is broken down into

$$\Phi_P = x_1 d_{P1} + x_2 d_{P2}$$

where d_{P1} and d_{P2} denote the excess performance of each of the two markets.

[14] See Poncet *et al.* (1996) and Chapter 22 of Copeland and Weston (1983).
[15] Cf. Grandin (1998), Jacquillat and Solnik (1987) and Broquet and van den Berg (1992).

With this method we can attribute the contribution of each market to the total performance of the portfolio. This in turn allows us to evaluate the manager's capacity to select the best-performing international securities and to invest in the most profitable markets.

McDonald's model only considers investments in stocks and represents international investment in the American market alone. However, the model can be generalised for the case of investment in several international markets, and for portfolios containing several asset classes. This is what Pogue *et al.* (1974) propose.

4.4.3 Pogue, Solnik and Rousselin's model

Pogue *et al.* (1974) also proposed an extension to the Jensen measure for international portfolios (see also Grandin, 1998, and Jacquillat and Solnik, 1987). Their model measures the performance of funds invested in French and international stocks, without any limit on the number of countries, and in French bonds. The model is written as follows:

$$R_{P_t} = \alpha_P + x_{OF,P}\beta_{OF,P}(I_{OF,t} - R_{Ft}) + x_{AF,P}\beta_{AF,P}(I_{AF,t} - R_{Ft}) + x_{WP}\beta_{WP}(I_{Wt} - R_{Wt}) + e_{Pt}$$

where

R_{Ft}	denotes the interest rate of the risk-free asset in the French market;
R_{Wt}	denotes the eurodollar rate;
$I_{OF,t}$, $I_{AF,t}$, $I_{W,t}$	denote the returns on the three representative indices: the French bond market index, the French stock market index and the worldwide stock market index for period t;
$x_{OF,P}$, $x_{AF,P}$ and x_{WP}	denote the proportion of the portfolio invested in each market;
$\beta_{OF,P}$, $\beta_{AF,P}$ and $\beta_{W,P}$	denote the systematic risk of each subset of the portfolio; and
α_P	denotes the portfolio's overall excess performance.

The result measures the manager's capacity to choose the most promising markets and his skill in selecting the best stocks in each market.

It is possible to go further into the analysis and breakdown of performance by using multi-factor models for international investment. These models will be presented in Chapter 6.

4.5 THE LIMITATIONS OF THE CAPM

4.5.1 Roll's criticism

Roll (1977) formulated a criticism of the CAPM. The core criticism relates to the fact that it is impossible to measure the true market portfolio.

Roll showed that the CAPM relationship implied that the market portfolio was efficient in the mean–variance sense. He deduced that to test the validity of the model, it was necessary to show that the market portfolio was efficient. However, the true market portfolio cannot be observed, because it must be comprised of all risky assets, including those that are not traded. Instead, we use a stock exchange index. The results of empirical tests are dependent on the index chosen as an approximation of the market portfolio. If this portfolio is efficient, then we conclude that the CAPM is valid. If not, we will conclude that the model is not valid. But these tests do not allow us to ascertain whether the true market portfolio is really efficient. Roll concludes from this that it is not possible to validate the CAPM empirically. This does not, nevertheless, mean that the model is not valid.

This criticism had consequences for the performance measurement models that were derived from the CAPM (Treynor and Jensen). If the index used as an approximation of the market portfolio is not efficient, then the portfolio performance result will depend on the index. By changing the index, the relative ranking of the portfolios is not necessarily maintained.

The fact that a portfolio which is not the true market portfolio is used leads to estimation errors in the betas. Some authors, such as Shanken (1987, 1992), present methods for correcting the measurement errors that are due to the fact that we are not observing the true market portfolio.

These criticisms have led to the development of other models. In the following chapter we shall successively present heteroskedastic models that enable better beta calculation and performance measurement models that do not depend on the market model. In Chapter 6 we shall present multi-factor models, which are mainly applied to portfolio risk analysis.

4.5.2 Conclusion

In spite of the criticism, the CAPM is widely appreciated as an asset valuation model. It has the advantage of being simple and is one of the best models for explaining returns. A consequence of the model was the development of passive management and index funds, the idea being that the best portfolio was the market portfolio. In the United States, Sharpe helped the firm Wells Fargo to set up its first index funds in the 1970s. The model gave rise to the first risk-adjusted performance measurement ratios. Among these, the Sharpe ratio is an indicator that is still widely used by the professionals.

However, without calling into question the contribution of the CAPM, the current consensus tends towards the idea that a single factor is not sufficient for explaining returns. Besides the market factor, two other factors have been identified: the size of the company and its book-to-market ratio. Fama and French[16] have carried out research on this subject. The firm Barra, for its part, has developed a more complete microeconomic model, which uses 13 fundamental factors. These factors are perfectly well known and defined, because they are directly linked to the securities (as such we speak of attributes), while in the case of the CAPM, the true theoretical market factor cannot be measured and must be approximated by a well-diversified market index. This approximation is one of the reasons for the criticism of the model formulated by Roll. On the basis of Roll's criticism, Ross proposed a multi-factor model that could be tested empirically. This model, while it presents the advantage over the Barra model of being based on an extension to the concepts of portfolio theory, proposes explaining the asset returns with the help of macroeconomic factors, without the theory specifying the number and nature of these factors, which makes it difficult to use. We will return to all of these models in detail in Chapter 6.

BIBLIOGRAPHY

Baker, III E.D., "Performance Basics: Simple in Theory, Hard to Apply – Part I", *Performance Measurement: Setting the Standards, Interpreting the Numbers*, CFA, 1991.

Bhattacharya, S. and Pfleiderer, P., "A Note on Performance Evaluation", *Technical Report* 714, Graduate School of Business, Stanford University, 1983.

[16] Fama and French have published several studies on the subject. See for example Fama and French (1995).

Black, F., "Capital Market Equilibrium with Restricted Borrowing", *Journal of Business*, no. 45, July 1972, pp. 444–455.

Breeden, D.T., "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities", *Journal of Financial Economics*, vol. 7, 1979, pp. 265–296.

Brennan, M., "Taxes, Market Valuation and Corporate Financial Policy", *National Tax Journal*, no. 25, 1970, pp. 417–427.

Briys, E. and Viala, P., *Eléments de théorie financière*, Nathan, 1995.

Broquet, C. and van den Berg, A., *Gestion de portefeuille – Actions, Obligations, Options*, 2nd edn, De Boeck Université, 1992.

Cobbaut, R., "Théorie du marché financier", in *Encyclopédie des marchés financiers*, Economica, 1997, pp. 1954–1978.

Copeland, T.E. and Weston, J.F., *Financial Theory and Corporate Policy*, 3rd edn, Addison-Wesley, 1988.

Dowd, K., "A Value at Risk Approach to Risk–Return Analysis", *Journal of Portfolio Management*, vol. 25, no. 4, summer 1999, pp. 60–67.

Elton, E.J. and Gruber, M.J., *Modern Portfolio Theory and Investment Analysis*, 5th edn, Wiley, 1995.

Fabozzi, F.J., *Investment Management*, Prentice Hall International Editions, 1995.

Fama, E., "Efficient Capital Markets: A Review of Theory and Empirical Work", *Journal of Finance*, vol. 25, no. 2, March 1970, pp. 383–417.

Fama, E.F. and French, K.R., "Size and Book-to-Market Factors in Earnings and Returns", *Journal of Finance*, vol. 50, no. 1, March 1995, pp. 131–155.

Farrell, Jr J.L., *Portfolio Management, Theory and Application*, McGraw-Hill, 2nd edn, 1997.

Gallais-Hamonno, G. and Grandin, P., "Le point sur... Les mesures de performance", *Banque & Marchés*, no. 42, September–October 1999.

Grandin, P., *Mesure de performance des fonds d'investissement, Méthodologie et résultats*, Economica, Gestion poche, 1998.

Grinblatt, M. and Titman, S., "Portfolio Performance Evaluation: Old Issues and New Insights", *Review of Financial Studies*, vol. 2, 1989, pp. 393–421.

Grinold, R.C. and Kahn, R.N., *Active Portfolio Management*, Irwin, 1995.

Hamon, J., "Efficience faible. Efficience semi-forte", in *Encyclopédie des Marchés Financiers*, Economica, 1997, pp. 408–432.

Henriksson, R.D., "Market Timing and Mutual Fund Performance: An Empirical Investigation", *Journal of Business*, vol. 57, no. 1, 1984, pp. 73–96.

Henriksson, R.D. and Merton, R.C., "On Market Timing and Investment Performance II: Statistical Procedures for Evaluating Forecasting Skills", *Journal of Business*, vol. 54, no. 4, 1981, pp. 513–533.

Jacquillat, B. and Solnik, B., *Les marchés financiers et la gestion de portefeuille*, 2nd edn, Dunod, 1987.

Jacquillat, B. and Solnik, B., *Marchés financiers, Gestion de portefeuille et des risques*, 3rd edn, Dunod, 1997.

Jensen, M.C., "The Performance of Mutual Funds in the Period 1945–1964", *Journal of Finance*, vol. 23, May 1968, pp. 389–419.

Jensen, M.C., "Capital Markets: Theory and Evidence", *The Bell Journal of Economics and Management Science*, vol. 3, no. 2, autumn 1972a.

Jensen, M.C., "Optimal Utilization of Market Forecasts and the Evaluation of Investment Performance", in *Mathematical Methods in Investment and Finance*, G.P. Szego and K. Shell, eds., Elsevier, 1972b.

Jensen, M., "Some Anomalous Evidence Regarding Market Efficiency", *Journal of Financial Economics*, vol. 6, no. 2/3, 1978, pp 95–101.

Korajczyk, R.A., *Asset Pricing and Portfolio Performance – Models, Strategy and Performance Metrics*, Risk Books, 1999.

Lhabitant, F.-S., "Portfolio Performance: Theoretical and Empirical Investigations on the Swiss Mutual Fund Market", Mémoire de Diplôme, Master en Banque et Finance, Université de Lausanne, Ecole des Hautes Etudes Commerciales, October 1994.

Lintner, J., "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets", *Review of Economics and Statistics*, February 1965, pp. 13–37.

Lintner, J., "The Aggregation of Investor's Diverse Judgements and Preferences in Purely Competitive Security Markets", *Journal of Financial and Quantitative Analysis*, vol. 4, no. 4, December 1969, pp. 347–400.

Lobosco, A., "Style/Risk-Adjusted Performance", *Journal of Portfolio Management*, vol. 25, no. 3, spring 1999, pp. 65–68.

McDonald, J., "French Mutual Fund Performance: Evaluation of Internationally Diversified Portfolios", *Journal of Finance*, vol. 28, no. 5, 1973, pp. 1161–1180.

Melnikoff, M., "Investment Performance Analysis for Investors", *Journal of Portfolio Management*, vol. 25, no. 1, fall 1998, pp. 95–107.

Merton, R.C., "An Intertemporal Capital Asset Pricing Model", *Econometrica*, vol. 41, September 1973, pp. 867–887.

Merton, R.C., "On Market Timing and Investment Performance I: An Equilibrium Theory of Value for Market Forecasts", *Journal of Business*, vol. 54, no. 3, 1981.

Miller, M.H., "The History of Finance: An Eyewitness Account", *Journal of Portfolio Management*, vol. 25, no. 4, summer 1999, pp. 95–101.

Modigliani, F. and Modigliani, L., "Risk-Adjusted Performance", *Journal of Portfolio Management*, winter 1997, pp. 45–54.

Mossin, J., "Equilibrium in a Capital Asset Market", *Econometrica*, October 1966, pp. 768–783.

Muralidhar, A.S., "Optimal Risk-Adjusted Portfolios with Multiple Managers", *Journal of Portfolio Management*, vol. 27, no. 3, spring 2001, pp. 97–104.

Philips, T.K., Rogers, G.T. and Capaldi, R.E., "Tactical Asset Allocation 1977–1994: The Long-Term Performance of TAA", *Journal of Portfolio Management*, fall 1996, pp. 57–64.

Pogue, G., Solnik, B. and Rousselin, A., "International Diversification: A Study of the French Mutual Funds", MIT, Working Paper, 1974.

Poncet, P., Portait, R. and Hayat, S., *Mathématiques financières: Evaluation des actifs et analyse du risque*, 2nd edn, Dalloz, 1996.

Roll, R., "A Critique of the Asset Pricing Theory's Tests", *Journal of Financial Economics*, March 1977, pp. 129–176.

Salvati, J., "Mesure de performance et gestion de portefeuille", in *Encyclopédie des marchés financiers*, Economica, 1997, pp. 1122–1139.

Shanken, J., "Multivariate Proxies and Asset Pricing Relations: Living with the Roll Critique", *Journal of Financial Economics*, vol. 18, 1987, pp. 91–110.

Shanken, J., "On the Estimation of Beta-Pricing Models", *Review of Financial Studies*, vol. 5, no. 1, 1992, pp. 1–33.

Sharpe, W.F., "A Simplified Model for Portfolio Analysis", *Management Science*, January 1963, pp. 277–293.

Sharpe, W.F., "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk", *Journal of Finance*, vol. 19, September 1964, pp. 425–442.

Sharpe, W.F., "Mutual Fund Performance", *Journal of Business*, January 1966, pp. 119–138.

Sharpe, W.F., *Investments*, 3rd edn, Prentice Hall, 1985.

Sharpe, W.F., "Asset Allocation: Management Style and Performance Measurement", *Journal of Portfolio Management*, vol. 18, winter 1992, pp. 7–19.

Sharpe, W.F., "The Sharpe Ratio", *Journal of Portfolio Management*, fall 1994.

Sharpe, W.F., "Morningstar's Risk-Adjusted Ratings", *Financial Analysts Journal*, July–August, 1998.

Simon, Y., *Encyclopédie des marchés financiers*, Economica, 1997.

Solnik, B., "The International Pricing of Risk: an Empirical Investigation of the World Capital Market Structure", *Journal of Finance*, May 1974b, pp. 365–378.

Solnik, B., "An International Market Model of Security Price Behavior", *Journal of Financial and Quantitative Analysis*, September 1974a, pp. 537–554.

Sortino, F.A. and Price, L.N., "Performance Measurement in a Downside Risk Framework", *Journal of Investing*, vol. 3, no. 3, fall 1994, pp. 59–64.

Taggart, Jr R.A., *Quantitative Analysis for Investment Management*, Prentice Hall, 1996.

Tobin, J., "Liquidity Preference as Behavior towards Risk", *Review of Economic Studies*, February 1958, pp. 65–85.

Treynor, J.L., "Toward a Theory of Market Value of Risky Assets", *Working Paper*, 1961, published in *Asset Pricing and Portfolio Performance – Models, Strategy and Performance Metrics*, Robert A. Korajczyk, ed., Risk Books, 1999.

Treynor, J.L., "How to Rate Management of Investment Funds", *Harvard Business Review*, vol. 43, January–February 1965, pp. 63–75.

Treynor, J.L. and Black, F., "How to Use Security Analysis to Improve Portfolio Selection", *Journal of Business*, vol. 46, no. 1, 1973, pp. 61–86.

Treynor, J. and Mazuy, K., "Can Mutual Funds Outguess the Market?", *Harvard Business Review*, vol. 44, July–August 1966, pp. 131–136.